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Equations of Electrogasdynamics and Applications.

Demetriades, A. and Hill, J. H.

FIRESTONE FLIGHT SCIENCES LABORATORY

GRADUATE AERONAUTICAL LABORATORIES
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

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EQUATIONS OF ELECTROGASDYNAMICS AND APPLICATIONS

by

Anthony Demetriades and Jack H. Hill

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I. INTRODUCTION

This report presents in summary work done in the period from 1959 to 1963 at the Hypersonic Research Project, Graduate Aeronautical Laboratories California Institute of Technology, on some theoretical aspects of the dynamics of ionized gases. The objective of this work was to set up a system of conservation equations when no magnetic field or complicated chemical effects are present. Emphasis is placed on the gross exchange processes among species at a point, rather than the gradient transport mechanisms (e. g., viscosity and heat conduction); in the past this point of view was often called "Gaseous Electronics". In subordinating the magnetic field and the chemistry to the dynamics it has been possible to explain a gratifying number of commonplace physical phenomena from first principles and to demonstrate the intimate connection between gaseous electronics and aerodynamics.

The conservation equations needed for a neutral-ion-electron mixture were derived with the aid of the elementary integral transport (Maxwell-Chapman) theory exactly as used for any ternary mixture. Departures made necessary because of the ionization include (a) the collisionless ("Vlasov") approximation, (b) a reformulation of the Chapman-Enskog procedure to include electric forces and (c) a convenient recourse to the inverse-fifth-power interparticle force law. Even so the resulting equations are merely a re-statement of the formidable problem of plasma physics. "Inviscid" equations are therefore written as counterparts of the inviscid (ideal) equations of aerodynamics. To illustrate where mathematical difficulties first appear, practical problems are solved with the aid of further approximations.

These problems, as well as the physical departures mentioned above, are relegated to Appendices.*

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* Individual Appendices can be obtained from Mrs. G. Van Gieson, Hypersonic Research Project Secretary, Firestone Flight Sciences Laboratory, California Institute of Technology, Pasadena, California.

II. EQUATIONS OF CONSERVATION

The integral transport theory of Maxwell and Chapman have been used to derive the specie and overall (global) conservation equations for a neutral-ion-electron mixture exactly as one derives these equations for any ternary mixture.* The customary two difficulties are the evaluation of the stresses and the evaluation of the collision terms.

II. 1. Validity of the Chapman-Enskog Procedure

In order to evaluate the stress terms, one needs some idea of the velocity distribution function. For small departures from equilibrium one usually obtains specie equations of the Navier-Stokes type. One classic test for such small departures is provided by the Chapman-Enskog theory.** In Appendix A this theory is re-written for a mixture of gases with body forces. A summary of Appendix A is as follows:

The non-dimensional form of the Maxwell-Boltzmann equation is

$$(\partial f_i/\partial t) + \vec{w}_i \cdot \nabla f_i + (\gamma/\varepsilon)\, \vec{F} \cdot \nabla_w f = (1/\varepsilon) \times (\text{collisions})$$

where

$$\varepsilon = \frac{\text{collision time}}{\text{flow time}} = (t_i/t)$$

and

$$\gamma = \frac{(\text{body force}) \times (\text{collision time})}{(\text{particle mass}) \times (\text{thermal velocity})} = (F_i t_i / m_i C_i) \quad .$$

When ε , γ $< < 1$ the following expansion is indicated:

$$f_i = f^{(0)} + \varepsilon f_1 + \gamma f_2 + \theta^{(0)} (\varepsilon^2 , \gamma^2) + \ldots\ldots$$

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* Background reading of Jeans[1] followed by Delcroix[2] is sufficient for this purpose.

** As carried out by Enskog and Hilbert, immortalized by Chapman and Cowling, interpreted by Lees and transcribed by Liu[3].

so that local thermalization requires a moderate body (field) force in addition to a rapid collision rate. Therefore the customary equations of ionized gases (and magnetohydrodynamics) are limited to the case where

$$\gamma \leq \epsilon \quad .$$

On manipulating this criterion for both ionized species, we find that in an ionized mixture of pressure p the critical field E_{cr} (above which the departures from equilibrium are significant) is

$$(E_{cr}/p) \sim (Q/q) \left[(\sqrt{\mu}\,)/(\mu + 1) \right] \quad ,$$

where q is the particle charge, Q the (elastic) collision cross-section and μ the mass ratio (e. g., electron to neutral). The implications of this simple criterion, spectacularly verified by the mobility experiments of Hornbeck[4], will be discussed in a later section under the title "Non-linear Effects".

II. 2. Collision Effects. "Collisionless-Collisional" Model

The second important point concerns the handling of the collision terms which for the ternary system considered consist of a matrix of nine collision terms. Multiple particle interactions are excluded.

Interactions between charged particles are excluded under the assumption that the collective, long-range electric forces prevail over the near collisions[5] ("Vlasov Approximation"). This criterion is expressible by[6]

Debye length $<<$ Macroscopic Scale

or*

$$h \equiv 17 \sqrt{(T/n)} << L \quad . \qquad \text{(T and n refer to the electrons.)}$$

Interactions between like particles are also excluded because of collision invariance, in case where the transport theory is not used to find the conservation of correlations (e. g., mutually orthogonal velocities).

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* The Debye length is about $7\sqrt{(T/n)}$ for plane geometries[7].

Interactions between charged particles and neutrals are assumed to be of the polarization type. In this way the happy circumstance that the collision integrals are integrable for inverse-fifth power force laws is utilized fully. This suggestion is an old one[2] and has been supported well by the experimental study of ionic mobilities[8]. Although both "Maxwellian molecules" and "Maxwellian ions" obey the inverse-fifth-power force law, the potential of the former is repulsive, while that of the latter attractive. This has caused some concern in the past, since attractive interaction potentials usually need a "hard-core" potential to prevent collapse of the two particles together. As shown in Appendix B, however, the (attractive) Maxwell ion potential generates centrifugal forces such that a hard core would be necessary only for the very low temperature encounters (order of a few degrees ^{o}K). The criterion for the validity of the inverse-fifth attractive force law is given approximately by

$$(p/q^2 v) > \left[(n_n^{7/3})/12\right]$$

where p is the over-all pressure, q the charge of the polarizing particle, and v and n_n the molecular volume and number density of the polarized material (e.g., the neutral particles).

The attractive form of the potential also necessitates a recomputation of the collision constants (symbolized A by Jeans[1]). For the momentum exchange integral, for instance, the integration constant was found to be

$$A_1 = 1.0214\ldots$$

Contrasted with that for neutral particles (Maxwell molecules):

$$A_1 = 2.6595\ldots$$

The last assumption made in this model is that the neutral-charged mean-free path λ_{cn} is much smaller than the charged-charged "path". In order of magnitude this criterion is expressible by

$$(h\Lambda/\ell n\Lambda) >> \lambda_{cn} ,$$

where h is the Debye distance and Λ the interaction distance between charged particles discussed by Spitzer[7].

In summary, we are proposing here a model "collisionless" in the Vlasov sense, but "collisional" so far as the interaction between charged and neutral species goes. Many plasma phenomena fall in this category, as the reader can verify by considering the above criteria.

II. 3. "Inviscid" Equations

In order to study certain problems in ionized gases, but also in order to demonstrate the connection between formulas used in gaseous electronics and formulas of fluid dynamics, the weak-field equations were written down when certain gradients were absent. The equations for each specie i then are as follows:

Continuity

$$\underbrace{(\partial n_i/\partial t)}_{C1} + \underbrace{\nabla \cdot n_i\vec{v}_i}_{C2} = \underbrace{\dot{n}_i}_{C3}$$

Momentum

$$\underbrace{\rho_i(d\vec{V}_i/dt)}_{M1} = \underbrace{n_i q_i \vec{E}_i}_{M2} - \underbrace{\nabla n_i kT_i}_{M3} - \underbrace{\dot{\rho}_i\vec{V}_i}_{M4} + \underbrace{\sum_j n_i n_j A_1 \pi_{ij}(\vec{v}_j-\vec{v}_i)}_{M5}$$

Energy

$$\underbrace{(\dot{n}_i/n_i)\left[(3kT_i/m_i) - |\vec{v}_i|^2\right]}_{E1} + \underbrace{(d/dt)(3kT_i/m_i)}_{E2} + \underbrace{2\,(kT_i/m_i)\nabla\cdot\vec{v}_i}_{E3}$$

$$= \underbrace{\sum_j (m_j/m_i)(A_1\pi_{ij}n_j)/(m_i+m_j)\left\{|v_j|^2 - |v_i|^2 + 3k\left[(T_j-T_i)/m_j\right]\right\}}_{E4} + \dot{Q}_i$$

In these equations k is Boltzman's constant and $\dot{n}$, $\dot{\rho}$ the number or mass production or destruction due to chemical events; $\dot{Q}_i$ is the

(positive or negative) amount of energy absorption or release because of such reactions. This is the extent to which inelastic events appear. Also, note that the body force does not appear explicitly in the energy equation, but only implicitly through the velocity $\vec{v}_i$.

The quantity π_{ij} is essentially an elastic collision cross-section for the interaction of specie i with specie j, and for the "Maxwellian ion" is

$$\pi_{ij} = \left[\frac{(D-1)q_i^2}{2\pi n_j} \quad \frac{m_i m_j}{m_i + m_j} \right]^{\frac{1}{2}} \qquad \text{(polarization)}$$

where D is the dielectric constant. The operator $(d/dt) \equiv (\partial/\partial t) + \vec{v}_i \cdot \nabla$.

It should be recognized that these inviscid equations include the mechanisms of gradient diffusion and electrical conductivity but not of viscosity and heat conduction. Gradient diffusion enters through the pressure gradient $\nabla n_i k T_i$ and electrical conductivity through the mutual "drag" between two different species. These equations should not be very useful in cases where friction dissipation and heat conduction (such as in an intense electric arc) are prevalent.

II. 3. 1. Recovery of Familiar Formulas (Gas Dynamics and Gaseous Electronics)

To test the applicability of the above equations, one tries to recover formulas familiar both in gas dynamics and gaseous electronics.

For a pure gas the terms M4 and M5 disappear and one recovers Euler's equation if no body forces act (M2 = 0). If body forces act ($m2 \neq 0$) their effect is the same as that of a pressure gradient in an inviscid gas. Hence the tendency of an ionized gas (one specie only) to accelerate outward under self-repulsion.

The use of a perfect gas law in relating pressure to the product $n_i k T_i$ provides a diffusion mechanism out of an Euler-type equation. We can recover Meyer's isothermal diffusion formula[1] by setting M1 = M2 = M4 = 0 and using $T_i = T$ = constant. The counterpart of $A\pi_{ij}$ in kinetic theory would be $m_i c_{o_i} Q_{ij}$, where Q_{ij} is an elastic collision

cross-section. Then

$$v_j - v_i = \frac{kT}{m_i c_i (n_j Q)} (\nabla n_i / n_i) \simeq (c_i \lambda_i / 3)(\nabla n_i / n_i) \quad .$$

The adiabatic law relating pressure to density can also be recovered from the energy equation (E1 = E4 = E5 = 0) with the aid of the momentum equations.

Famíliar formulas of gaseous electronics are also obtained easily. The simple Ohm's law (velocity proportional to the field) obtains by M1 = M3 = M4 = 0. Observe therefore that this often-misused formula holds for an inert, inertialess and diffusionless gas only. If $A_1 \pi_{ij}$ is equal to $m_i c_{o_i} Q_{ij}$, then simple mobility ideas hold; this is "Langevin's first law"[8]. When $A_1 \pi_{ij}$ is obtained from polarization considerations, M2 and M5 give "Langevin's second law"[8]. The ambipolar diffusion idea[9] is obtainable from M2, M3, and M5 for an isothermal gas using electron and ion momentum conservation (the danger of the usual assumptions of ambipolar diffusion is self-evident, however).

"Sheath" formulas are obtainable easily from combinations such as M1 = M3 = M4 = 0 or M3 = M4 = M5 = 0; the latter gives the familiar collisionless "Child's Law"[9].

The classic Boltzmann distribution[9] of diffusing particles in a force field can be obtained directly from M2 and M3.

It should be emphasized that although when the critical electric field is above the limit given earlier the energy equation should always be used, the inverse is not true; in other words, it should be possible to have temperature differences (although small) between the species but still use the "weak field" momentum equation.

II. 3. 2. Problems in Electrogasdynamics

II. 3. 2. 1. Flow of Charged Particles in a Two-Dimensional Channel (Appendix C)

As an example, it was attempted to solve the problem of a charged stream moving normal to the plane formed by the axes of two infinite parallel wires across which a voltage difference exists.

Consider the steady flow of a fluid carrying charged particles. If the velocity field of the charges is related to the local electric field intensity by the mobility concept, then

$$\nabla \cdot \left[\overset{+}{n} (\vec{u} + \overset{+}{K}\vec{E}) \right] = 0$$

$$\nabla \cdot \left[\overset{-}{n} (\vec{u} + \overset{-}{K}\vec{E}) \right] = 0 \qquad \text{(conservation of particles)}$$

$$\nabla \cdot \vec{E} = (1/\varepsilon)(\overset{+}{e}\overset{+}{n} - \overset{-}{e}\overset{-}{n}) \qquad \text{(Poisson's equation)}$$

$$\nabla \times \vec{E} = 0$$

where

$\vec{u}$ = velocity vector of neutral gas

$\overset{+}{n}$, $\overset{-}{n}$ = number densities of + and - charges

$\overset{+}{K}$, $\overset{-}{K}$ = mobilities of + and - charges

$\overset{+}{e}$, $\overset{-}{e}$ = charge per particle for + and - charges

In Appendix C the effect of the motion of the charges upon the motion of the neutral gas is ignored, so that $\vec{u}$ is assumed to be known. Also, the effect of $\overset{+}{n}$, $\overset{-}{n}$ upon $\vec{E}$ (Poisson's equation) is ignored. Thus $\vec{u}$ is known from ordinary fluid mechanics and $\vec{E}$ is known from electrostatics. The non-diemsnional problem parameter is

$$\alpha = (kV/U\ell\ell_{nk}^2)$$

where V is the voltage drop between the wires and

$$k \equiv \frac{1 + \left[1 - (r^2/\lambda^2)\right]^{\frac{1}{2}}}{(r/\lambda)} \quad .$$

The results show that when α is high the flow is dominated by the electric field, so that the charged particles follow the electrostatic field lines. When α is small ($\alpha \longrightarrow 0$) the particles travel along straight paths (along $\vec{U}$). For $\alpha \leq 0.55$ the "communication corridor" between the two wires "breaks away", i. e., for such α's particles leaving one wire cannot arrive at the other but are dragged downstream by the flow. This is therefore a rudimentary "discharge blowout" criterion.

The above results were extended to the two-dimensional flow in a channel containing two semi-cylindrical electrodes. Although the problem still remains uncoupled (in the sense that $\overset{+}{n} = 0$ and the applied electric field is undisturbed) the neutral flow field was allowed to vary

as in potential flow. In this way (since $\nabla \cdot \vec{v} = \nabla \cdot \vec{E} = 0$) it became possible to use a streamfunction method for mapping the field; the parameters in the problem are

$$\alpha \equiv \frac{KV}{\lambda U \ell n (\pi^2 r^2 / 16 \lambda^2)} \qquad \text{and} \qquad (r/\lambda) \quad ,$$

where K is the mobility, V the inter-electrode potential, λ the distance between poles, U the flow velocity and r the electrode radius.

Maps have been drawn for $\alpha = 0.25$, 0.5, and 1.0 and $(r/\lambda) = 0.5$. These also show that the "blowout" value of α occurs between 0.5 and 1. The "blowout voltage" is then proportional to the mass-flow flux ρU and the interelectrode separation as expected.

II. 3. 2. 2. Transient Space Charge Problem

Another problem examined was the unsteady one-dimensional flow of charges in a parallel-plate gap. This problem, described by

$$(\partial n^{\pm} / \partial t) + \nabla \cdot n^{\pm} \vec{v}^{\pm} = 0$$

$$\vec{v}^{\pm} = K^{\pm} \vec{E}$$

$$\nabla \cdot \vec{E} = s (n^{+} - n^{-})$$

$$\nabla \times \vec{E} = 0$$

$$\int_0^d E \, dx = V_o$$

$$n^{\pm} (t = 0) = n_o$$

has interested plasma physicists on its own merits. The non-dimensional parameters here are

$$s \equiv (4\pi q n_o d^2 / v_o)$$

$$\alpha = (K^{+} / K^{-}) \quad .$$

This problem has been traditionally attacked in the limit of $s \longrightarrow 0$, i. e., by linearizing to the case where the induced field is negligible. For non-negligible space charge (induced field) one might say that because of differences in mobility the ions remain stationary ($\alpha = 0$) while the electrons evacuate the gap. An exact solution in closed form for $s < 2$ (weak to moderate space charge) is then obtained from Reference 10 (Appendix D); for $s > 2$ the problem can only be solved with a computer (see below). After the electrons have left the gap only the positive specie remains. An exact closed-form solution for this motion was found by Morrison and Edelson[11] for $s < 2$ and by Demetriades[10] for arbitrary s (Appendix D).

Recently Hill (Appendix E) obtained numerical solutions to this problem for arbitrary s and α with the aid of an IBM 7090 computer. His results corroborate the above analytical solutions and provide a basis for the inclusion of a great variety of reactions (i. e., a non-zero r. h. s. to the continuity equation).

III. NON-LINEAR EFFECTS

As mentioned earlier the assumption of a "weak" electric field is crucial to the insensitivity of the charged-particle mobility (and temperature) to the electric field strength. When the latter exceeds its critical value self- and mutual equilibration of the species is impossible. The formulation of the transport equations for such strong fields is equivalent to the formulation of a transport theory well beyond the Navier-Stokes approximation.

Efforts have been made by various workers to predict at least some transport processes in this "non-linear" regime. As shown by simple considerations[4] when the critical field for a certain specie is exceeded, that specie begins drifting with a speed proportional to $E^{\frac{1}{2}}$ (not E) and its temperature begins deviating from that of the overall mixture* Many observed phenomena can be explained by these simple considerations.

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* The voluminous literature on "Electron Heating" deals with just that phenomenon.

As seen from Figure 1 of Reference 4 the Mach number (v_e/c_e) of the electrons of a neutral-ion-electron mixture subjected to a strong field is always much smaller than unity. For ions, however, the Mach number (v_i/c_i) is of order one, so that the formation of ion "fronts" is in principle possible on this precept. Ion "waves" have indeed been observed experimentally.

The ability of the lighter particles (electrons) to heat up very readily under an electric field supplies a good understanding of a very familiar phenomenon, i. e., dielectric breakdown of gases. It also supplies a ready explanation of why it is that the electrons, rather than the positive ions, are responsible for the breakdown process. Very elementary ideas predict that avalanche ionization occurs (roughly) when a charged particle, moving in the direction of the field, acquires between two successive collisions energy equal to the ionization potential. But ions and electrons carry the same charge, and their mean-free-paths are close enough so that on that basis their capability to ionize by collision should be about the same. The non-linear theory simply states, however, that because of their smaller mass the electrons retain sufficient energy during a great number of collisions so that when the crucial time comes they need only a small additional contribution from the field in order to acquire ionization energy.

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APPENDIX A

CHAPMAN-ENSKOG PROCEDURE FOR A MIXTURE INCLUDING BODY FORCES

A. General

It is our eventual objective to write the conservation laws for an ionized mixture for the case of slight deviation of the velocity distribution function of each specie from the Maxwell distribution. We begin with the Maxwell-Boltzmann equation. For a specie i colliding with other species, j = 1, 2, ... :

$$(\partial f_i/\partial t) + \vec{w}_i \cdot \nabla_{r_i} f_i + (\vec{F}_i/m_i) \cdot \nabla_{w_i} f_i = \sum_j \int_{w_j} \int_{p} \int_{\epsilon} (f_i' f_j' - f_i f_j)\, v_{ij}\, p\, dp\, d\epsilon\, dw_j \qquad (1)$$

where

f_i = distribution function of i

r_i = configuration coordinates

w_i = total instantaneous velocity of members of i

F_i = total instantaneous force on i between collisions (a field force)

m_i = mass of members of i

f_j = distribution function of j

$( )'$ = quantity after a collision i - j

v_{ij} = total relative velocity i - j during a collision

p = impact parameter, ϵ = impact angle } see Jeans[1], p. 216

w_j = total instantaneous velocity of members of j.

Let us non-dimensionalize (1) by using

$$t = \bar{t}\,\tilde{t} \qquad x = c_{o_i}\bar{t}\,\tilde{x} \qquad p = \sigma_o \tilde{p} \qquad F = F_o \tilde{F}$$

$$w_i = c_{o_i}\tilde{w} \qquad f = f_o \tilde{f} \qquad v_{ij} = c_o \tilde{v}_{ij}$$

where

$\bar{t}$ = a macroscopic time constant for every specie

c_{o_i} = average thermal velocity of the i specie

$f_{o_i} \equiv \dfrac{n_i}{c_{o_i}^3}$

σ_o = effective collision diameter for i and j

F_o = a certain macroscopic force (field force)

($\tilde{\nabla}$ is non-dimensional).

Then eq. (1) becomes

$$\frac{f_{o_i}}{\bar{t}}\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \left(\frac{\vec{c}_{o_i} f_{o_i}}{c_{o_i}\bar{t}}\right)\cdot\tilde{\nabla}_r \tilde{f}_i + \left(\frac{F_o}{m_i}\frac{f_{o_i}}{c_{o_i}}\right)\tilde{F}\cdot\tilde{\nabla}_w \tilde{f}_i = \tag{2}$$

$$= f_{o_i}\sum_j c_{o_i}\sigma_o^2 n_j \iiint (\tilde{f}_i'\tilde{f}_j' - \tilde{f}_i\tilde{f}_j)\tilde{\rho}\,d\check{\rho}\,d\check{\epsilon}\,\tilde{v}_{ij}\,d\tilde{w}_j$$

Multiplying through by $\bar{t}/f_{o_i}$ we get

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{w}_i\cdot\tilde{\nabla}_r\tilde{f}_i + \left(\frac{F_o}{m_i}\frac{\bar{t}}{c_{o_i}}\right)\tilde{F}_i\cdot\tilde{\nabla}_w\tilde{f}_i = \sum_j (\bar{t}\,c_{o_i}\sigma_o^2 n_j)\iiint(\dots)\dots d\tilde{w}_j \tag{3}$$

From Maxwellian kinetic theory we know that the mean-free time τ_i of the specie i and its mean free path λ_i can be connected by'

$$\tau_i = \frac{\lambda_i}{c_{o_i}} \simeq \frac{1}{\sum_j c_{o_i}\sigma_{o_{ij}}^2 n_j\sqrt{1+\frac{m_i}{m_j}}}$$

Therefore,

$$\bar{t}\,c_{o_i}\sigma_o^2 n_j = \bar{t}\,c_{o_i}\sigma_o^2 n_j\frac{\sqrt{1+m_i/m_j}}{\sqrt{1+m_i/m_j}} = \frac{\bar{t}}{\tau_{ij}}\frac{1}{\sqrt{1+\frac{m_i}{m_j}}}$$

Therefore,

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{w}_i\cdot\tilde{\nabla}\tilde{f}_i + \frac{F_o}{m_i}\frac{\tau_i}{c_{o_i}}\frac{\bar{t}}{\tau_i}\tilde{F}_i\cdot\tilde{\nabla}_w\tilde{f}_i = \sum_j \frac{1}{\varepsilon_{ij}\sqrt{\quad}}\iiint\dots d\tilde{w}_j \tag{4}$$

with

$$\frac{1}{\varepsilon_{ij}} \equiv \frac{\bar{t}}{\tau_{ij}} \equiv \bar{t}\,\sigma_{o_{ij}}^2 c_{o_i} n_j\sqrt{1+\frac{m_i}{m_j}}$$

Finally, if we define,

$$\varepsilon_i \equiv \frac{\tau_i}{\bar{t}} \qquad \left(\tau_i \equiv \frac{1}{\sum_j \frac{1}{\tau_{ij}}},\ \varepsilon_i = \frac{1}{\sum_j \frac{1}{\varepsilon_{ij}}}\right)$$

$$\gamma_i \equiv \frac{F_o\tau_i}{m_i c_{o_i}}$$

we have

$$\frac{\partial f_i}{\partial \tilde{t}} + \tilde{w}_i \cdot \tilde{\nabla} \tilde{f}_i + \left(\frac{\partial_i}{\varepsilon_i}\right) \tilde{F} \cdot \tilde{\nabla}_w \tilde{f}_i = \sum_j \frac{1}{\varepsilon_{ij} \sqrt{1 + \frac{m_i}{m_j}}} \iint \cdots d\tilde{w}_j \qquad (5)$$

B. Example: Pure Gas, No Body Forces

Consider first, a pure gas without body forces, i.e., $\gamma_i = 0$.

Then $\varepsilon_{ij} \equiv \varepsilon_i$ and, omitting the subscript,

$$\varepsilon \left(\frac{\partial \tilde{f}}{\partial \tilde{t}} + \tilde{w} \cdot \tilde{\nabla} \tilde{f} \right) = \iiint \left(\tilde{f}'_i \tilde{f}'_j - \tilde{f}_i \tilde{f}_j \right) \cdots d\tilde{w}_j \qquad (6)$$

Now if $\tilde{f}$ is Maxwellian, both sides of eq. (6) vanish (Cf. Jeans, p. 210) since f is independent of the spatial coordinates. However, if the various parameters in $\tilde{f}$ (e.g., the temperature, mean velocity, etc) depend on $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, say, then the r.h.s. of (6) vanishes but the l.h.s. does not, generally. Consequently, the Maxwellian distribution cannot be used to describe a gas where transport processes occur.

When $\tilde{f}$, however, is a weak function of the $\tilde{t}$ and $\tilde{r}$, which is tantamount to saying that ε is small, we expect that f will vary little from the Maxwellian. The Hilbert-Chapman-Enskog procedure consists of expanding

$$f = f_0 + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots\cdots$$

for

$$\varepsilon << 1 \text{ only.}$$

In our non-dimensionalization, re-write the above as

$$f = f_0 \tilde{f} = f_0 \left(\tilde{f}^{(0)} + \varepsilon \tilde{f}^{(1)} + \varepsilon^2 \tilde{f}^{(2)} + \cdots\cdots \right) .$$

The l.h.s. of (6) is

$$\varepsilon \left[\left(\frac{\partial \tilde{f}^{(0)}}{\partial \tilde{t}} + \tilde{w} \cdot \tilde{\nabla}_r \tilde{f}^{(0)} \right) + \varepsilon \left(\frac{\partial \tilde{f}^{(1)}}{\partial \tilde{t}} + \tilde{\nabla}_r \cdot \tilde{f}^{(1)} \right) + \cdots \right]$$

The r.h.s. is

$$\iiint_{w\,p\,\epsilon} \left(\tilde{f}'^{,2} - \tilde{f}^2 \right) \tilde{\rho} \, d\tilde{p} \, d\epsilon \, v \, d\tilde{w}$$

The primed f represents the post-collision function:

$$f' = \tilde{f}'^{(0)} + \varepsilon \tilde{f}'^{(1)} + \varepsilon^2 \tilde{f}'^{(2)} + \cdots\cdots .$$

Therefore the collision integral reads

$$\iiint \left(\tilde{f}^{(0)} + 2\varepsilon \tilde{f}^{(1)} \tilde{f}^{(0)} + \dots - \tilde{f}^{(0)} - 2\varepsilon \tilde{f}^{(1)} \tilde{f}^{(0)} - \dots \right) \tilde{p}\, d\tilde{p}\, dt\, \tilde{r}\, d\tilde{w}$$

The complete equation to the first order is, then

$$\frac{\partial \tilde{f}^{(0)}}{\partial \tilde{t}} + \tilde{w} \cdot \tilde{\nabla} \tilde{f}^{(0)} = \iiint \tilde{f}^{(0)} \left(\tilde{f}^{(1)\prime} - \tilde{f}^{(1)} \right) \tilde{p}\, d\tilde{p}\, dt\, \tilde{r}\, d\tilde{w}$$

This formula can be used to solve for $\tilde{f}^{(1)}$. In the Chapman-Maxwell theory it can be used as a "generatrix" for deriving the first-order conservation equations.

C. Example II: Gaseous Mixture with No Body Forces

In this case $\gamma = 0$ again (Cf. eq. (5)) but let us say there are several species present, so that the mean collision time of each member i with some j is τ_{ij}. The definitions of τ_i, ε_i, and ε_{ij} given previously are then in order. Equation (5) then is

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{w}_i \cdot \tilde{\nabla} \tilde{f}_i = \sum_j \frac{1}{\varepsilon_{ij} \sqrt{1 + \frac{m_i}{m_j}}} \iiint \left(\tilde{f}_i' \tilde{f}_j' - \tilde{f}_i \tilde{f}_j \right) \dots d\tilde{w}_j$$

The correct expansion of f_i should now be in terms of a small parameter ε_i which compounds the effects of all types of collisions for each specie i:

$$\varepsilon_i \equiv \frac{\tau_i}{\tilde{t}} = \frac{1}{\tilde{t} \sum_j \sigma_{ij} \delta_{ij}^2 n_j}$$

as defined previously. Therefore,

$$\tilde{f}_i = \tilde{f}_i^{(0)} + \varepsilon_i \tilde{f}_i^{(1)} + \varepsilon_i^2 \tilde{f}_i^{(2)} + \dots\dots ,$$

and similarly for j, with ε_j. Also,

$$\tilde{f}_i' = \tilde{f}_i^{(0)} + \varepsilon_i \tilde{f}^{(1)\prime} + \varepsilon_i^2 \tilde{f}^{(2)\prime} + \dots\dots .$$

D. Case (i): Specie Masses and Cross-Sections Nearly Uniform, Temperature Uniform

In order to insert the above expansions into eq. (5) for a mixture, it is first necessary to compare the magnitudes of ε for each specie, a problem which did not arise in the case of the pure gas. If our collision cross-sections and masses in the mixture are of comparable magnitude, we have: (for equal temperatures $T_1 = T_2 = T_3 = \ldots$)

$$c_{o_1} = c_{o_2} = c_{o_3} = \ldots \equiv c_o$$

Setting all numerical factors (including the cross-sections) equal to unity for convenience, we obtain

$$\tau_{11} = \frac{1}{c_o n_1} = \tau_{21} = \tau_{31} = \ldots$$

$$\tau_{12} = \tau_{22} = \tau_{32} = \ldots$$

$$\tau_{13} = \tau_{23} = \tau_{33} = \ldots$$

Thus,

$$\tau_1 = \frac{1}{\frac{1}{\tau_{11}} + \frac{1}{\tau_{12}} + \frac{1}{\tau_{13}}} = \frac{1}{\frac{1}{\tau_{21}} + \frac{1}{\tau_{22}} + \frac{1}{\tau_{23}}} = \tau_2 = \tau_3 = \ldots$$

Therefore,

$$\varepsilon_1 \equiv \frac{\tau_1}{\bar{t}} = \varepsilon_2 = \varepsilon_3 = \ldots$$

We have thus proved that the small parameters ε are of comparable magnitude when the masses and cross-sections are comparable, (and temperatures) regardless of the relative species abundance.

In inserting the expanded functions into (5) we now are justified in retaining the same number of terms (to order ε in this case) for each f_i:

$$\left(\frac{\partial \tilde{f}_i^{(0)}}{\partial \tilde{t}} + \tilde{w}_i \cdot \tilde{\nabla}_r \tilde{f}_i^{(0)} \right) + \varepsilon_i \left(\frac{\partial \tilde{f}_i^{(1)}}{\partial \tilde{t}} + \tilde{w}_i \cdot \tilde{\nabla} \tilde{f}_i^{(1)} \right) + \ldots$$

$$= \sum_j \frac{1}{\varepsilon_{ij}} \iiint \left[\tilde{f}_i^{(0)} \tilde{f}_j^{(0)} + \varepsilon_i \tilde{f}_j^{(0)} \tilde{f}_i^{(1)} + \varepsilon_j f_j^{(1)} \tilde{f}_i^{(0)} + \ldots - \right.$$

$$\left. - \tilde{f}_i^{(0)} \tilde{f}_j^{(0)} - \varepsilon_i \tilde{f}_i^{(1)} \tilde{f}_j^{(0)} - \varepsilon_j \tilde{f}_i^{(1)} \tilde{f}^{(0)} - \ldots \right] \tilde{\rho} \, d\tilde{p} \, d\tilde{t} \, \tilde{v}_{ij} \, d\tilde{w}_j$$

or

$$\mathcal{D}\tilde{f}_i^{(0)} \equiv \frac{\partial \tilde{f}_i^{(0)}}{\partial t} + \tilde{w}_i \cdot \nabla_r \tilde{f}_i^{(0)} = \sum_j \frac{1}{\varepsilon_{ij}} \iiint \left(\varepsilon_j \tilde{f}_i^{\prime(0)} \tilde{f}_j^{\prime(1)} + \varepsilon_i \tilde{f}_i^{(1)} \tilde{f}_j^{\prime(0)} - \ldots \right) \tilde{p}' d\tilde{p}' \ldots d\tilde{w}_j$$

Since

$$\tilde{f}_i^{\prime(0)} = \tilde{f}_i^{(0)}$$

(but note $f_{o_i} \neq f_{o_j}$ generally)

we obtain

$$\mathcal{D}\tilde{f}_i^{(0)} = \sum_j \frac{1}{\varepsilon_{ij}} \iiint \tilde{f}_i^{(0)} \tilde{f}_j^{(0)} \left(\varepsilon_j \frac{\tilde{f}_j^{\prime(1)}}{\tilde{f}_j^{\prime(0)}} + \varepsilon_i \frac{\tilde{f}_i^{\prime(1)}}{\tilde{f}_i^{\prime(0)}} - \varepsilon_j \frac{\tilde{f}_j^{(1)}}{\tilde{f}_j^{(0)}} - \varepsilon_i \frac{\tilde{f}_i^{(1)}}{\tilde{f}_i^{(0)}} \right) \tilde{p} d\tilde{p} \tilde{v}_{ij} d\epsilon d\tilde{w}_j$$

as the equation giving first-order approximation to the distribution function.

As an illustration, we will work out the case of a ternary mixture containing one abundant and two rare species:

$$n_1 >> n_2, n_3 \qquad T_1 \simeq T_2 \simeq T_3$$

$$m_1 \simeq m_2 \simeq m_3 \; ; \qquad c_{o_1} = c_{o_2} = c_{o_3} \equiv c_o \; .$$

We make the following table:

$$\tau_{11} = \frac{1}{c_o n_1} \qquad \tau_{21} = \tau_{11} \qquad \tau_{31} = \tau_{11}$$

$$\tau_{12} = \frac{1}{c_o n_2} = \frac{n_1}{n_2} \tau_{11} \qquad \tau_{22} = \frac{n_1}{n_2} \tau_{11} \qquad \tau_{32} = \frac{n_1}{n_2} \tau_{11}$$

$$\tau_{13} = \frac{1}{c_o n_3} = \frac{n_1}{n_3} \tau_{11} \qquad \tau_{23} = \frac{n_1}{n_3} \tau_{11} \qquad \tau_{33} = \frac{n_1}{n_3} \tau_{11}$$

Therefore,

$$\tau_1 = \tau_{11}$$

$$\tau_2 = \tau_{11}$$

$$\tau_3 = \tau_{11}$$

so that

$$\varepsilon_1 = \varepsilon_{11} = \varepsilon_2 = \varepsilon_3$$

(also proved on page 5).

Then,

$$\mathcal{D}\tilde{f}_1^{(0)} \equiv \left(\frac{\partial}{\partial \tilde{t}} + \tilde{w}_1 \cdot \tilde{\nabla}_r\right)\tilde{f}_1^{(0)}$$

$$= \tilde{f}_1^{(0)}\left\{\frac{\varepsilon_1}{\varepsilon_{11}}\iiint \tilde{f}_1^{(0)} 2\left(\frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}}\right)\tilde{\rho}\, d\tilde{\rho}\, d\epsilon\, \tilde{v}_{11}\, d\tilde{w}_1 + \frac{1}{\varepsilon_{12}}\iiint \tilde{f}_2^{(0)}\left(\varepsilon_2 \frac{\tilde{f}_2^{\prime(1)}}{\tilde{f}_2^{(0)}} + \varepsilon_1 \frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \right.\right.$$

$$\left.\left. - \varepsilon_2 \frac{\tilde{f}_2^{(1)}}{\tilde{f}_2^{(0)}} - \varepsilon_1 \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}}\right)\tilde{\rho}\, d\tilde{\rho}\, d\epsilon\, \tilde{v}_{12}\, d\tilde{w}_2 + \frac{1}{\varepsilon_{13}}\iiint \tilde{f}_3^{(0)}\left(\varepsilon_3 \frac{\tilde{f}_3^{\prime(1)}}{\tilde{f}_3^{(0)}} + \varepsilon_1 \frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \varepsilon_3 \frac{\tilde{f}_3^{(1)}}{\tilde{f}_3^{(0)}}\right.\right.$$

$$\left.\left. - \varepsilon_1 \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}}\right)\tilde{\rho}\, d\tilde{\rho}\, d\epsilon\, \tilde{v}_{13}\, d\tilde{w}_3\right\}$$

But from

$$\frac{\varepsilon_1}{\varepsilon_{11}} = 1 \qquad \frac{\varepsilon_2}{\varepsilon_{12}} = \frac{\varepsilon_1}{\varepsilon_{12}} << 1$$

$$\frac{\varepsilon_1}{\varepsilon_{12}} = \frac{\varepsilon_{11}}{\varepsilon_{12}} << 1 \qquad \frac{\varepsilon_3}{\varepsilon_{13}} = \frac{\varepsilon_{11}}{\varepsilon_{13}} << 1$$

$$\varepsilon_1/\varepsilon_{13} = \varepsilon_{11}/\varepsilon_{13} << 1$$

We get

$$\mathcal{D}\tilde{f}_1^{(0)} = 2\tilde{f}_1^{(0)}\iiint\left(\tilde{f}_1^{\prime(1)} - \tilde{f}_1^{(1)}\right)\tilde{\rho}\, d\tilde{\rho}\, d\epsilon\, \tilde{v}_{11}\, d\tilde{w}_1$$

This formula shows that in the first approximation, the correction to the distribution function of the abundant specie are independent of the presence of the rare species. (Also, the transport coefficients of the abundant specie are similarly independent of the rare species.)

We can also form the equations for $f_2^{(0)}$ and $f_3^{(0)}$. For the former:

$$\mathcal{D}\tilde{f}_2^{(0)} = \tilde{f}_2^{(0)}\sum_j \frac{1}{\varepsilon_{2j}}\iiint \tilde{f}_j^{(0)}\left(\varepsilon_j \frac{\tilde{f}_j^{\prime(1)}}{\tilde{f}_j^{(0)}} + \varepsilon_2 \frac{\tilde{f}_2^{\prime(1)}}{\tilde{f}_2^{(0)}} - \varepsilon_j \frac{\tilde{f}_j^{(1)}}{\tilde{f}_j^{(0)}} - \varepsilon_2 \frac{\tilde{f}_2^{(1)}}{\tilde{f}_2^{(0)}}\right)\ldots d\tilde{w}_j$$

Now

$$\frac{\varepsilon_2}{\varepsilon_{21}} = \frac{\varepsilon_{11}}{\varepsilon_{11}} = 1 \qquad \frac{\varepsilon_2}{\varepsilon_{22}} = \frac{\varepsilon_{11}}{\varepsilon_{22}} << 1$$

$$\frac{\varepsilon_2}{\varepsilon_{23}} = \frac{\varepsilon_{11}}{\varepsilon_{23}} << 1$$

Consequently,

$$\delta \tilde{f}_2^{(0)} = \tilde{f}_2^{(0)} \iiint \tilde{f}_1^{(0)} \left(\frac{\tilde{f}_1'^{(1)}}{\tilde{f}_1^{(0)}} + \frac{\tilde{f}_2'^{(1)}}{\tilde{f}_2^{(0)}} - \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} - \frac{\tilde{f}_2^{(1)}}{\tilde{f}_2^{(0)}} \right) \tilde{\rho} \, d\tilde{\rho} \, d\epsilon \, \tilde{v}_{12} \, d\tilde{\kappa}_1$$

In the same manner, we get the expression for $\delta \tilde{f}_3^{(0)}$.

These two formulas state that the corrections to the distribution function of the rare species are dominated by the presence and behavior of the abundant specie only. (Also, the transport coefficients of one of the rare species are dominated by the characteristics of the abundant specie and are independent of the characteristics of the other rare specie. (Cf. Jeans, p. 220 ff.))

E. Case (ii): One of the Species is Very Light, Temperatures Equal

In the above example we saw that for a ternary mixture in which the masses and temperatures (and the cross-sections) are comparable, the perturbation of the distribution function of the abundant specie is unaffected by the presence of the rare species; the perturbation to the d.f. of the rare species, on the other hand, depend exclusively on the abundant specie.

Consider, now the case where one of the rare species consists of very light particles:

$$m_1 \; ; \; m_2 >> m_3 \; ; \; (m_3/m_1) = (m_3/m_2) \equiv \mu$$

while

$$n_1 >> n_2 \, , \, n_3 \quad ,$$

with the collision cross-sections σ_0^2 remaining comparable. From our general formulae:

$$\tau_{ij} = \frac{1}{\sigma_0 \, n_j \sqrt{1 + \frac{m_i}{m_j}}}$$

(We put $\sigma_0^2 \simeq 1$ since we are making an ordering procedure.)

$$\varepsilon_{ij} \equiv \frac{\tau_{ij}}{\bar{t}}$$

$$\varepsilon_i = \left[\sum_j \frac{1}{\varepsilon_{ij}} \right]^{-1}$$

We form a table of the pertinent quantities:

$$\tau_{11} = \frac{1}{c_{0,1} n_1 \sqrt{2}} \qquad \tau_{21} = \tau_{11} \qquad \tau_{31} = \sqrt{\mu}\,\tau_{11}$$

$$\tau_{12} = \frac{n_1}{n_2}\tau_{11} \qquad \tau_{22} = \frac{n_1}{n_2}\tau_{11} \qquad \tau_{32} = \frac{n_1}{n_2}\sqrt{\mu}\,\tau_{11}$$

$$\tau_{13} = \frac{n_1}{n_3}\sqrt{\mu}\,\tau_{11} \qquad \tau_{23} = \frac{n_1}{n_3}\sqrt{\mu}\,\tau_{11} \qquad \tau_{33} = \frac{n_1}{n_3}\sqrt{\mu}\,\tau_{11}$$

Note (as shown on the last column) that the shortest time is τ_{31}, that is the collision time for specie 3 (the very light one) with the abundant, heavy specie 1.

The table for the ε_{ij}'s is similar to the above, when ε is substituted for τ everywhere. For the parameters ε we get:

$$\varepsilon_1 = \varepsilon_{11}$$

$$\varepsilon_2 = \varepsilon_{11}$$

$$\varepsilon_3 = \sqrt{\mu}\,\varepsilon_{11}$$

but only if now $\frac{n_3}{n_1}\frac{1}{\sqrt{\mu}} << 1$, instead of $\frac{n_3}{n_1} << 1$ as before, for temperatures being equal. In this case, the characteristic small parameter for the two heavy species (1 and 2) is given by the collision time of specie 1 with itself. The characteristic parameter for the light specie, however, is much smaller than ε_1 and ε_2 (by a factor $\sqrt{\mu}$) reflecting its much faster collision rate.

Since $\varepsilon_3 << \varepsilon_1, \varepsilon_2$ the consistent expansion is

$$\tilde{f}_1 = \tilde{f}_1^{(0)} + \varepsilon_1 \tilde{f}_1^{(1)}$$

$$\tilde{f}_2 = \tilde{f}_2^{(0)} + \varepsilon_2 \tilde{f}_2^{(1)}$$

$$\tilde{f}_3 = \tilde{f}_3^{(0)}$$

where we have facitly assumed that $\tilde{f}_3^{(1)}$ is also much smaller than unity. This means that due to its greater (by a factor $1/\sqrt{\mu}$) collision rate, specie 3 "equilibrates" with itself much faster than do species 1 and 2 equilibrate with themselves. Parenthetically, note that this is not meant to say, for example, that in a mixture of neutrals, ions and electrons the electrons are more likely to fe found "in equilibrium" than the heavier species. We will later see that in such a mixture, which has also just undergone some disquieting process, the ions tend to return to a Maxwellian equilibrium very slowly, but their mean energy (temperature) tends to that of the overall gas very rapidly; the electrons behave in exactly the opposite way. Finally it should be kept in mind that these results are true for $(n_3/n_1)(1/\sqrt{\mu}) << 1$ only.

For specie 1 we get, again

$$\partial \tilde{f}_1^{(0)} = \tilde{f}_1^{(0)} \frac{1}{\varepsilon_{12}} \iiint \tilde{f}_2^{(0)} \left(\varepsilon_2 \frac{\tilde{f}_2^{\prime(1)}}{\tilde{f}_2^{(0)}} + \varepsilon_1 \frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \varepsilon_2 \frac{\tilde{f}_2^{(1)}}{\tilde{f}_2^{(0)}} - \varepsilon_1 \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} \right) \tilde{\rho}' d\tilde{\rho}' d\epsilon + \tilde{\nu}_{12} d\tilde{W}_2$$

$$+ 2 \tilde{f}_1^{(0)} \frac{1}{\varepsilon_{11}} \iiint \tilde{f}_1^{(0)} \left(\frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} \right) \tilde{\rho}' d\tilde{\rho}' d\epsilon + \tilde{\nu}_{11} d\tilde{W}_1$$

$$+ \tilde{f}_1^{(0)} \frac{\sqrt{\mu}}{\varepsilon_{13}} \iiint \tilde{f}_3^{(0)} \left(\varepsilon_1 \frac{\tilde{f}_1^{\prime(1)}}{\tilde{f}_1^{(0)}} - \varepsilon_1 \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} \right) \tilde{\rho}' d\tilde{\rho}' d\epsilon + \tilde{\nu}_{13} d\tilde{W}_3$$

and since

$$\frac{\varepsilon_2}{\varepsilon_{12}} = \frac{n_2}{n_1} << 1 \qquad \frac{\varepsilon_1}{\varepsilon_{11}} = 1$$

$$\frac{\varepsilon_2}{\varepsilon_{11}} = \frac{n_2}{n_1} << 1 \qquad \frac{\varepsilon_1 \sqrt{\mu}}{\varepsilon_{13}} = \frac{\varepsilon_1 \sqrt{\mu}}{n_3 \sqrt{T_1} \varepsilon_1} = \frac{n_3}{n_1} << 1$$

Again we see that the second integral is the only survivor, meaning that again the abundant specie is modified only by itself.

An identical calculation for specie 2 shows, as before, that its distribution function is perturbed by collisions with members of 1 only.

F. The Light Specie Is of Elevated Temperature

Say now that the light specie is elevated in temperature so that it is hotter than species 1 and 2 by a factor ϕ:

$$m_3 c_{o_3}^2 = \phi (m_1 c_{o_1}^2) = \phi (m_2 c_{o_2}^2)$$

Then if the collision cross-sections are still comparable so that they can be set equal to unity, the new table of τ_{ij}'s is:

$$\tau_{11} = \frac{1}{c_{o_1} n_1 \sqrt{2}} \qquad \tau_{21} = \tau_{11} \qquad \tau_{31} = \frac{\sqrt{\mu}}{\sqrt{\phi}} \tau_{11}$$

$$\tau_{12} = \frac{n_1}{n_2} \tau_{11} \qquad \tau_{22} = \frac{n_1}{n_2} \tau_{11} \qquad \tau_{32} = \frac{n_1}{n_2} \frac{\sqrt{\mu}}{\sqrt{\phi}} \tau_{11}$$

$$\tau_{13} = \frac{n_1}{n_3} \sqrt{\mu} \tau_{11} \qquad \tau_{23} = \frac{n_1}{n_3} \sqrt{\mu} \tau_{11} \qquad \tau_{33} = \frac{n_1}{n_3} \frac{\tau_{11}}{\sqrt{\phi}}$$

Thus again

$$\varepsilon_1 = \varepsilon_{11}$$

$$\varepsilon_2 = \varepsilon_{11}$$

$$\varepsilon_3 = \sqrt{\frac{\mu}{\theta}}\ \varepsilon_{11}$$

So that the effect of the increased temperature of specie 3 is to increase its thermalization ability, as expected.

G. Mixture with Body Forces

When the body forces are present,

$$\delta_i \equiv \frac{\tau_i F_i}{n_i c_{0i}} \neq 0$$

and the equation (5) is pertinent:

$$\frac{\partial \tilde{f}_i}{\partial \tilde{t}} + \tilde{w}_i \cdot \tilde{\nabla}_r \tilde{f}_i + \frac{\lambda_i}{\varepsilon_i} \tilde{F} \cdot \tilde{\nabla}_w \tilde{f}_i =$$

$$= \sum_j \frac{1}{\varepsilon_{ij}\sqrt{1 + m_i/m_j}} \iiint \left(\tilde{f}_i' \tilde{f}_j' - \tilde{f}_i \tilde{f}_j \right) \tilde{p}\, dp + \tilde{v}_{ij} \cdot \tilde{w}_i$$

We saw, earlier, that the Chapman-Enskog procedure is valid when the left-hand side of this equation is at most of order unity. This requires that

$$(\gamma_i/\varepsilon_i) \leq 1 \quad ,$$

and, in fact, when $\gamma_i = \varepsilon_i$ a double expansion is not necessary. In this case our criterion for γ_i becomes

$$\gamma_i = \varepsilon_i$$

or

$$\frac{F_i \bar{t}}{m_i c_{0i}} = 1$$

But $m_i c_{0i}^2$ = constant = W_T (thermal energy) and thus

$$\frac{F_i \bar{t}}{m_i c_{0i}} = \frac{F_i \bar{t} c_{0i}}{W_T} = \frac{F_i \bar{t}}{\sqrt{m_i}\sqrt{W_T}} = 1$$

Since $\bar{t}$ and W_T are constant for the mixture, it follows that the body force F_i which makes γ_i small depends on the mass of the particles i in question.

When γ_i is thus of order ε_i we can expand the distribution functions in a series containing ε_i only. The results are identical with those found in the previous section, except that now the l. h. s. of the first-order equation contains the force term:

$$\mathcal{D} \tilde{f}_i^{(0)} \rightarrow \left(\frac{\partial}{\partial \bar{t}} + \tilde{w}_i \cdot \tilde{\nabla}_r + \tilde{F} \cdot \tilde{\nabla}_w \right) \tilde{f}_i^{(0)}$$

H. Weakly Ionized Mixture

We have by now developed sufficient tools to consider a weakly ionized mixture from the standpoint of the Chapman-Enskog approximation. This mixture resembles the case of the termary mixture with one very abundant and one very light component. We will subscribe the neutrals by 1, the ions by 2, and the electrons by 3:

$n_1 >> n_2, n_3$ ($n_2 = n_3$ usually)

$m_1, m_2 >> m_3$ ($m_1 = m_2$ usually)

It can be correctly argued that the Maxwellian expression for the mean free path

$$\lambda_i = \frac{1}{\sum_j G_{0ij}^2 n_j \sqrt{1 + \frac{m_i}{m_j}}}$$

is inapplicable for the Coulomb-dominated charged-charged encounters. However, taking the expression for λ_i to be valid will still give an expression agreeing with out Vlasov-approximated Maxwell-Boltzman equation for each specie (see Section I) on the right-hand-side of the equation.

So far as the l. h. s. of the equations go, note that

$$\gamma_1 = \frac{F_1 \bar{t}}{m_1 c_{o_1}} = 0 \qquad \text{since } F_1 = 0$$

where F is an electromagnetic force. Also

$$\gamma_2 = \frac{F_2 \bar{t}}{m_2 c_{o_2}}$$

from our requirement $\gamma_i \simeq \mathcal{O}(\varepsilon_i)$. Then

$$\frac{F_2}{m_2 c_{o_2}} = \frac{1}{\bar{t}} << \frac{1}{\tau_1}$$

from our requirement $\tau_1 / \bar{t} << 1$. Thus,

$$\frac{F_2}{m_2 c_{o_2}} << \frac{c_{o_1}}{\lambda_1}$$

or

$$\frac{F_2 \lambda_1}{m_2 c_{o_2}^2} \frac{c_{o_2}}{c_{o_1}} \doteq \frac{F_2 \lambda_1}{W_T} \sqrt{\frac{m_1}{m_2}} = \frac{F_2 \lambda_1}{W_T} << 1$$

For a charged-neutral interaction the Maxwellian expression for the mean-free-path is approximately valid, hence

$$\lambda_1 \simeq \frac{1}{n_1 \sigma_0^2}$$

Hence

$$\frac{F_2}{n_1 W_T \sigma_0^2} \doteq \boxed{\frac{F_2}{p \sigma_0^2} << 1}$$

where p is the pressure of the gas, since from Dalton's law

$$p \equiv \sum_K p_K \simeq p_1$$

For the electrons, $\frac{m_1}{m_2} \doteq \frac{1}{\mu}$ and from

$$\frac{F_3 \lambda_1}{W_T} \sqrt{\frac{m_1}{m_3}} = \frac{F_3}{W_T} \frac{\lambda_1}{\sqrt{\mu}}$$

We get

$$\boxed{\frac{F_3}{p\sigma_0^2} \ll \sqrt{\mu}}$$

Therefore, the force F_3 necessary to make the electrons depart from a Maxwellian distribution is much smaller than that necessary to make the ions depart from the same distribution. Our final results are:

$$\frac{1}{\tilde{f}_1^{(0)}}\left(\frac{\partial}{\partial \tilde{t}} + \tilde{w}_1 \cdot \tilde{\nabla}_r\right) \tilde{f}_1^{(0)} = 2 \iiint (\tilde{f}_1'^{(1)} - \tilde{f}_1^{(1)})\, \tilde{\sigma}\, d\tilde{\varphi}\, d\tilde{\epsilon}\, d\tilde{w}_1$$

$$\frac{1}{\tilde{f}_2^{(0)}}\left(\frac{\partial}{\partial \tilde{t}} + \tilde{w}_2 \cdot \tilde{\nabla}_r + \tilde{F}_2 \cdot \tilde{\nabla}_w\right) \tilde{f}_2^{(0)} = \iiint \tilde{f}_1^{(0)} \left(\frac{\tilde{f}_1'^{(1)}}{\tilde{f}_1^{(0)}} + \frac{\tilde{f}_2'^{(1)}}{\tilde{f}_2^{(0)}} - \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} - \frac{\tilde{f}_2^{(1)}}{\tilde{f}_2^{(0)}}\right) \cdots d\tilde{w}_1$$

$$\frac{1}{\tilde{f}_3^{(0)}}\left(\frac{\partial}{\partial \tilde{t}} + \tilde{w}_3 \cdot \tilde{\nabla}_r + \tilde{f}_3 \cdot \tilde{\nabla}_w\right) \tilde{f}_3^{(0)} = \iiint \tilde{f}_1^{(0)} \left(\frac{\tilde{f}_1'^{(1)}}{\tilde{f}_1^{(0)}} + \frac{\tilde{f}_3'^{(1)}}{\tilde{f}_3^{(0)}} - \frac{\tilde{f}_1^{(1)}}{\tilde{f}_1^{(0)}} - \frac{\tilde{f}_3^{(1)}}{\tilde{f}_3^{(0)}}\right) \cdots d\tilde{w}_1$$

and the necessary criteria are

$$\frac{T_i}{t} \ll 1 \qquad (i = 1, 2, 3)$$

$$\frac{n_3}{n_1} \frac{1}{\sqrt{\mu}} \ll 1$$

$$\frac{F_{2,3}}{p\sigma_0^2} \ll \sqrt{\mu_{2,3}}$$

APPENDIX B

COMPUTATION OF MOMENTUM EXCHANGE INTEGRAL FOR INVERSE 5TH ATTRACTIVE FORCES

1. Introduction

It has been shown that the classical collision integral featuring interaction forces of the central type can be simply evaluated only when the interparticle force varies as the inverse fifth power of the interparticle distance. Assuming that the collision of two neutral particles obey just such a law, Maxwell formulated his famous "Maxwellian molecule" theory in which the magnitudes of the inter-specie momentum and energy exchange cross-sections were essentially computed. Although the inverse-fifth repulsive force assumed is still too long-ranged to be realistic, the tutorial value of Maxwell's calculation is undeniable.

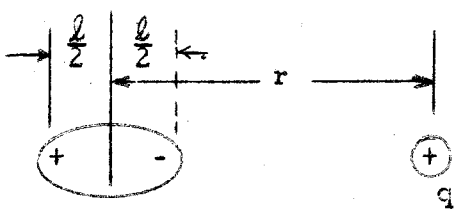
One type of inter-particle encounter for which the inverse-fifth force law is justifiable on physical grounds is that between a charged and a polariseable neutral particle. It has been therefore often suggested that this force law be used to describe the behavior of charged particles in a neutral gas. Langevin carried out this calculation for the momentum exchange between ions and neutrals and managed to predict the mobility of the former with a fair amount of success. No such calculation has been made for the electrons, however. The reason for this is that the motion and temperature of the latter exhibit the familiar non-linear dependence on the electric field strength, in which case the entire concept behind Langevin's calculation is not applicable.

The calculation of the effective momentum exchange cross-section based on the polarization will be worked out below for ions and electrons so that, in spite of the shortcomings realized long ago by Maxwell himself, an orderly comparison between the behavior of ions and electrons can be made. The numerical constant multiplying the momentum exchange expression turns out to be much lower than its counterpart for the repulsive force. Also, the polarization mode of momentum exchange appears to need no "hard-core" for all but the lowest temperatures of inter-particle encounter. These details will be presented below.

2. Polarization Force Between Charged and Neutral Particles

Consider a charged particle of charge q at some distance r away from a polarizable molecule of size l.

For $r >> l$,



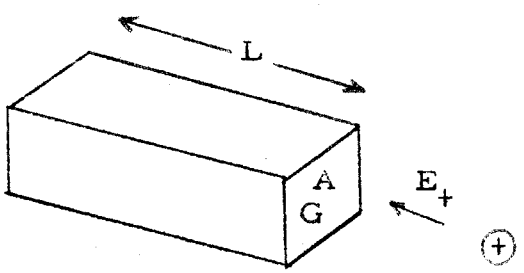
$$F_+ = \frac{q^2}{(r-\frac{l}{2})^2} - \frac{q^2}{(r+\frac{l}{2})^2} = q^2\left[\frac{r^2+rl-\frac{l^2}{4}-r^2+rl-\frac{l^2}{4}}{(r^2-\frac{l^2}{4})^2}\right] \qquad \text{(B-1)}$$

$$= 2\mu\,(q/r^3)$$

where

$$\mu \equiv l\,q = \text{dipole moment of molecule.} \qquad \text{(B-2)}$$

It remains to calculate $\mu = \mu(r)$. To do this, consider a block of matter (containing molecules) under the action of the field $\vec{E}$ of the ion. If the material is polarizeable, the total field E within it is



$$E = E_+ + E' = E_+ + 4\pi\sigma = DE_+ \qquad \text{(B-3)}$$

where D is the dielectric constant. Also,

$$DE_+ = 4\pi\sigma \qquad \text{(B-4)}$$

where σ is the surface charge on each face of area A. Thus

$$\mu \equiv \frac{\text{total momentum of block}}{\text{particle density}} = \frac{\sigma AL}{ALN} = \frac{\sigma}{N} = \frac{E_+\,(1/4\pi)\,(D-1)}{N} \qquad \text{(B-5)}$$

$$= E_+\,\frac{D-1}{4\pi N}$$

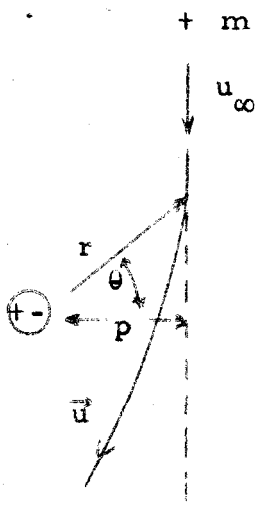
and since $E_+ = (q/r^2)$, we get

$$F_+ = 2\,(q^2/r^5)\left[(D-1)/4\pi N\right] \qquad \text{(B-6)}$$

3. Dynamics of the Inverse-Fifth-Power Attractive Force

Having determined that the polarization force is an inverse-fifth-power force of the central type, we will describe the dynamics of the interaction as the charged particle (projectile) approaches an assumedly stationary neutral molecule or atom (target).

For this purpose we will use the familiar "equivalent potential" formulation. The total energy of the projectile is conserved:



$$E = \text{const.} = \tfrac{1}{2} mv^2 + V(r) = \tfrac{1}{2} mv_\infty^2 \qquad (B-7)$$

where v is the instantaneous velocity of the projectile and V its instantaneous potential energy in the (polarization) field of the neutral; v_∞ is the projectile velocity at infinity.

The angular momentum of the projectile is also conserved:

$$\ell \equiv \text{ang. mom.} = mr^2\dot{\theta} = mpv_\infty \quad . \qquad (B-8)$$

Also,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 \quad . \qquad (B-9)$$

Therefore,

$$E = \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \qquad (B-10)$$

$$= \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} mr^2 \left[(p^2 v_\infty^2)/r^4\right] + V(r) \qquad (B-11)$$

$$= \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} \left[(mp^2 v_\infty^2)/r^2\right] + V(r) \quad . \qquad (B-12)$$

The energy of the projectile is thus reduced solely in terms of r; in particular, the first term above is the kinetic energy along the interparticle radius; the last two terms are only functions of r and can be lumped into an "equivalent potential".

$$V'(r) = \tfrac{1}{2} \left[(mp^2 v_\infty^2)/r^2\right] + V(r) \quad . \qquad (B-13)$$

The potential V(r) is, of course, the previously derived polarization potential:

$$V(r) = K/(4r^4) \tag{B-14}$$

where

$$K = -\frac{q^2(D-1)}{2\pi N} < 0 \tag{B-15}$$

It is now clear that when curves of V' and E are plotted versus r, the difference between these two is exactly the kinetic energy $\frac{1}{2}mr^2$. In Figure B. 2 an example is shown for a case where V represents a repulsive potential (V > 0). Hence V' is also everywhere positive. On the other hand, E, being everywhere constant, intercepts the V' curve at the point of closest approach, where $\dot{r}$ is, of course, zero.

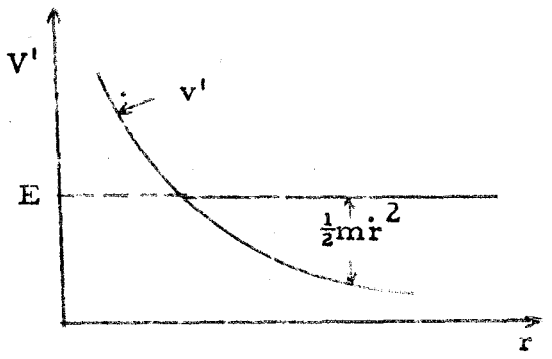


Figure B. 2

Returning to the inverse-fifth attractive potential, we write it in the non-dimensional form:

$$\frac{2V'(r)}{mp^2v_\infty^2} = (1/r^2) + \left(\frac{K}{2mp^2v_\infty^2}\right)(1/r^4) \tag{B-16}$$

a maximum occurring at

$$r^* = \left(\left|\frac{K}{2mp^2v_\infty^2}\right|\right)^{\frac{1}{2}}, \quad V^*/(mp^2v_\infty^2) = (mp^2v_\infty^2/K). \tag{B-17}$$

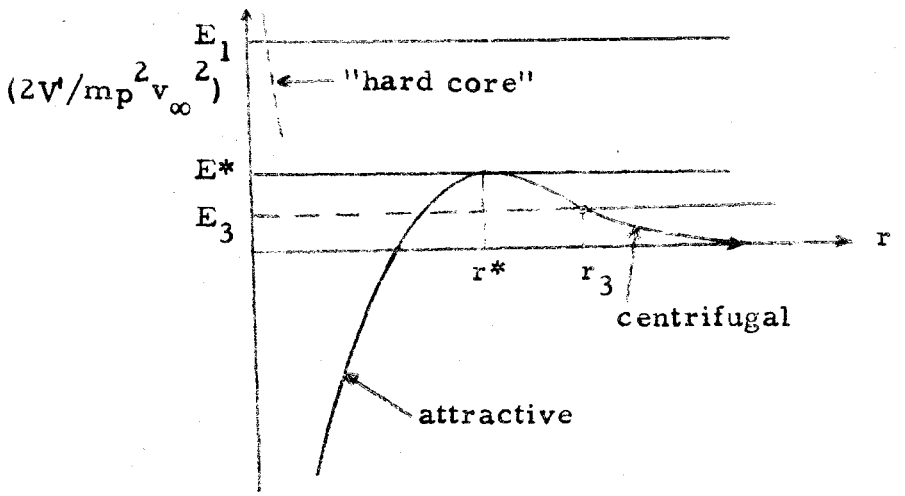


Figure B. 3

In Figure B. 3 we see that the centrifugal force (first term in equation B-16)) dominates over the polarization force (second term) at the far distances. Three cases E_1 , E_2 , and E_3 of interaction energy are shown. When the projectile charged particle has energy E_3 , its distance of closest approach is r_3 . For energies such as E_1 the projectile can penetrate the neutral target. One then needs to postulate (or derive with quantum theory) a "hard-core" correction to the theory in a manner analogous to the Lennard-Jones correction. Orbits for such interactions are shown by Hirchfelder, Curtiss, and Byrd.

Contrary to a first impression, the high-energy encounters belong to levels <u>below</u> E* in Figure B. 3 ; that is, $E_3 > E^* > E_1$. This can be seen as follows: the energy of encounter is $\frac{1}{2} mv_\infty^2$. The "hump" of the potential, however goes as $V'^* \sim (mv_\infty^2)^2$ (from equation (B-17)). Thus as the energy of encounter E increases, V'* grows much faster than E and the orbit is of the E_3 type. It is therefore clear that if this was to be the case for most encounters between charged particles and neutrals, the inverse-fifth-power force law would be sufficient and no "hard-core" postulates would be necessary. Now, this criterion can be written as

$$E_\infty < V'^* \qquad \text{(B-18)}$$

or, using equation (B-17)

$$\tfrac{1}{2} mv_\infty^2 < (1/4)\,(mv_\infty^2)^2 \quad (4p^4/K) \quad . \qquad \text{(B-19)}$$

But $4p^4/K$ is the potential energy due to polarization at the impact distance, therefore when

$$\boxed{\text{(kinetic energy)} > 4\ \text{(polarization potential)}_p} \qquad \text{(B-20)}$$

the inverse-fifth representation is sufficient. If some estimate of an average impact parameter p could be made (as will be attempted below) then this criterion can be written in terms of macroscopic quantities.

4. Numerical Estimate of the Validity of the Criterion

The above criterion states that the kinetic energy of the projectile should at all times be greater than 4 times the potential energy of the polarization potential at an impact parameter away. Now the kinetic energy of the projectile will be of the order of the thermal energy of the charged particle; and if we agree that the minimum temperature of the charged particles is the temperature of the over-all fluid, we have

$$\tfrac{1}{2}\mu v_e^2 \simeq (3/2)\,KT \qquad \left[\text{Note: } \mu_{en} = \frac{m_e m_n}{m_e+m_n} \doteq m_e \quad ; \quad \mu_{in} = \frac{m_i m_n}{m_i+m_n} \doteq (m_i/2)\right] .$$

(e: electron; e : ion ; n : neutral). (B-21)

By an argument similar to that given to explain Langevin's second theory of ionic mobility we say that

$$\text{P. E. at } p = \frac{e^2(D-1)}{8\pi n_n p^4} \qquad (B-22)$$

where e is the electronic charge, D the dielectric constant and n_n the density of the neutrals. Thus, from our criterion,

$$\frac{\text{K. E.}}{\text{P. E.}} = 4\,\frac{(3/2)\,KT 8\pi n_n p^4}{e^2(D-1)} = \frac{48\,P\pi p^4}{e^2(D-1)} > 1 \quad , \qquad (B-23)$$

where P is the over-all pressure of the gas. Also, we take

$$D - 1 \simeq 4\pi n a^3 \qquad (B-24)$$

where a^3 is of the order of the molecular volume or 10^{-24} cm^3. Thus,

$$\frac{\text{K. E.}}{\text{P. E.}} = \frac{48\,P\pi p^4}{e^2 4\pi n_n a^3} = \frac{12 P p^4}{e^2 n_n a^3} \quad . \qquad (B-25)$$

Now p is of order $n_n^{-1/3}$, thus $p^4 = n_n^{-4/3}$ and

$$\boxed{(p/e^2a^3) \gtrdot (\ln n_n/12)^{7/3}} \quad . \qquad (B-26)$$

This criterion can best be presented in a P vs. n_n plot as shown on Figure B. 4. Within the assumptions made we see that the inverse-fifth-power law for charged-neutral encounters holds for the pressure-density combinations of usual interest.

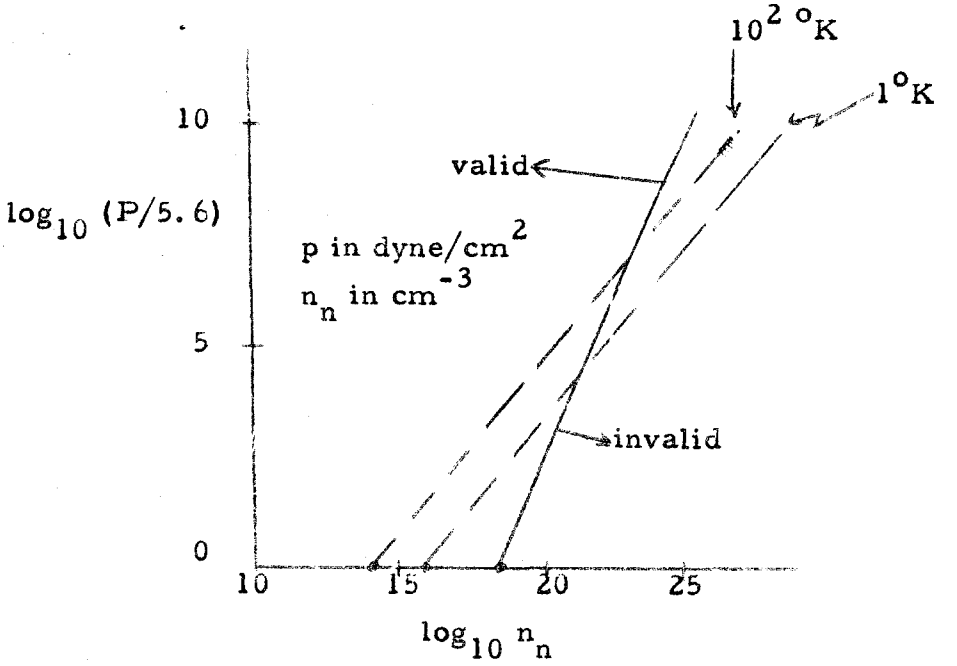


Figure B. 4

5. Calculation of Charged-Neutral Momentum Exchange

Jeans has shown that when a particular specie (unsubscribed in the nomenclature which follows) exchanges momentum by colliding with any number of other species j, then the momentum exchanged per unit time is given by the integral

$$\sum_j \sqrt{K \mu_j}\ A_1 \int_w \int_{w_j} (w - w_j)\, f\, f_j\, dw\, dw_j \quad , \tag{B-27}$$

when the particle-particle interactions obey an inverse-fifth-power central force law. Here μ_j has the same significance as before, that is,

$$\mu_j \equiv mm_j/(m+m_j) \quad , \tag{B-28}$$

while K is given by equation (B-15), that is

$$K = \frac{(D-1)_j q^2}{2\pi n_j} \quad . \tag{B-29}$$

In the general case K is whatever constant of proportionality appears in the expression for the central force.

There remains to interpret A_1. This has been worked out by Maxwell to be

$$A_1 \equiv 4\pi \int_a \cos(2\theta'/2)\, a\,da \quad , \tag{B-30}$$

where a and θ' (we retain Jean's notation) are geometrical parameters of the orbit -- in our case the orbit of the electron or ion about the polarized neutral molecule. In particular, a is related to the ratio of the kinetic energy of the interaction (based on the reduced mass μ and the velocity v_∞ of the projectile far away from the target) to the potential energy of the projectile at a distance away from the target equal to the collision parameter p:

$$a \equiv p\,(\mu v^2/K)^{1/4} \quad . \tag{B-31}$$

That is,

$$a^4 = \tfrac{1}{2}\ \frac{K.E.}{P.E.} \quad . \tag{B-32}$$

Finally, θ' is the usual apsidal angle of the orbit, and as such depends on a in a complicated implicit fashion.

Maxwell calculated A_1 for the repulsive inverse fifth potential for which the orbits are quite simple. As shown by Figure B.2, all such orbits have a distance of minimum approach, and since all values of the interaction kinetic energy are legitimate in this context, the limits of a in equation (B-30) are 0 and ∞. He found

$$A_1 = 4\pi \int_0^\infty \cos^2(\theta'/2)\, a\,da = 2.6595\ldots\ldots \quad . \tag{B-33}$$

In the case of the (attractive) polization force, however, we saw that those interactions not meeting the requirement [equation (B-20)] cannot be included in an integral such as A_1 (Hirschfelder, Curtiss, and Byrd

show that such orbits are quite complicated and not consistent with the definition of A_1). Further, by discussing equation (B-26) the conclusion was drawn that these excluded orbits do not, under certain circumstances, contribute much to the overall momentum lost by the specie in question. If the particles executing these orbits are ignored, the limits of a for the polarization potential can be found by combining equations (B-20) and (B-32):

$$\sqrt[4]{2} < a < \infty \quad . \tag{B-34}$$

Then*,

$$A_1 = 4\pi \int_{\sqrt[4]{2}}^{\infty} \cos^2(\vartheta'/2)\, a\, da = 1.01 \ldots\ldots \quad . \tag{B-35}$$

Returning now to the momentum exchange integral [equation (B-17)] we find that

$$\begin{pmatrix}\text{momentum lost}\\ \text{by the specie i per}\\ \text{unit time and volume}\end{pmatrix} = A_1 n_i \sum_j \sqrt{K\mu}\ (\vec{V}-\vec{V}_j)\, n_j \quad . \tag{B-36}$$

In the text of this paper we have used the nomenclature

$$\pi_{ij} = \sqrt{K\mu}\ \left[\text{dimensions} : \frac{(\text{mass})(\text{length})^3}{\text{time}}\right] . \tag{B-37}$$

6. Electron-Neutral and Ion-Neutral Momentum Exchange

We conclude this discussion with a practical illustration of the significance of the momentum exchange term. On the basis of the above, the momentum equation for the electrons, excluding shear terms, is

$$\rho_e (d\vec{v}_e/dt) = n_e \vec{F}_e - \nabla n_e kT_e - \dot{\rho}_e \vec{V}_e + n_e n_n \pi_{en} (\vec{v}_e - \vec{v}_n) \quad , \tag{B-38}$$

where n refers to the neutrals; similarly, for the ions'

$$\rho_i (d\vec{v}_i/dt) = n_i \vec{F}_i - \nabla n_i kT_i - \dot{\rho}_i \vec{v}_i + n_i n_n \pi_{in} (\vec{v}_i - \vec{v}_n) \quad . \tag{B-39}$$

The last term in each of these equations is, as shown, equal to the "drag" exerted on the motion of each charged specie by the neutrals. For singly

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* This integral was computed numerically by T. van Harreveld.

charged ions, K for the ions and electrons is the same; therefore, the difference in π_{in} and π_{en} arises from μ . For example,

$$(\pi_{in}/\pi_{en}) = \sqrt{(\mu_{in}/\mu_{en})} = \left[\frac{\dfrac{m_i m_n}{m_i + m_n}}{\dfrac{m_e m_n}{m_e + m_n}} \right]^{\frac{1}{2}} \simeq \left[\frac{(m_n/2)}{m_e} \right]^{\frac{1}{2}} \tag{B-40}$$

$$= (m_n/2m_e)^{\frac{1}{2}} .$$

Typically the mass of the neutral is $m_n = 10^4 m_e$, and thus

$$(\pi_{in}/\pi_{en}) \sim 100 \tag{B-41}$$

so that the ions are dragged considerably more than electrons and thus possess much lower mobility.

Now, since π obviously includes a collision cross-section (in this instance of the polarization type) one seeks to reconcile the momentum exchange term as derived here [Cf. equation (B-36)] with what one could obtain from elementary ideas. For even without going through the arguments elaborated here, one could write, say, for the electrons

$$\begin{bmatrix} \text{momentum exchanged per unit time and volume} \end{bmatrix} = \begin{bmatrix} \text{momentum exchanged by one electron per collision} \end{bmatrix} \begin{bmatrix} \text{collisions per unit time} \end{bmatrix} \begin{bmatrix} \text{electrons per unit volume} \end{bmatrix}$$

and in the usual nomenclature,

$$\Delta(\text{momentum}) = m_e (\vec{v}_e - \vec{v}_n)\, \nu\, n_e \qquad (\nu = \text{frequency})$$

so that if

$$\nu = (c_e/\lambda_e) \doteq c_e\, n_n\, Q_{en}$$

where c_e is the electron thermal velocity, λ_e its m. f. p. , and Q_{en} the electron-neutral elastic cross-section, then

$$\Delta(\text{momentum}) = m_e c_e n_e n_n Q_{en} (\vec{v}_e - \vec{v}_i) \quad . \qquad \text{(B-42)}$$

Obviously, the right hand side of equations (B-36) and (B-42) should be equivalent, so that $\mathcal{T}_{en}$ corresponds to $m_e c_e Q_{en}$ and has been, in fact, used in this sense in the text (Cf. page 7). This naive thinking also gives a relationship corresponding to equation (B-40), found for the polarization force:

$$(\mathcal{T}_{in}/\mathcal{T}_{en}) = \frac{m_i c_i Q_{in}}{m_e c_e Q_{en}} \sim (m_i c_i / m_e c_e) = (m_i/m_e)\sqrt{(m_e/m_i)} = \sqrt{(m_i/m_e)}$$

when $m_i c_i^2 = m_e c_e^2$ (at thermal equilibrium) which is a condition implied throughout this paper by the "weak field" approximation (Appendix A).

APPENDIX C

CHARGED PARTICLE TRAJECTORIES

FOR THE CASE WHERE THE INDUCED FIELD IS NEGLIGIBLE

1. Equations of Motion

Consider a neutral gas flowing through a region of space in which there is an externally imposed electric field present. Suppose that at some instant (t = 0) a charged particle (of charge e) is created at a point (x_o , y_o , z_o). What is the subsequent history of such a particle?

If the inertia of the particle is ignored and if the "mobility" concept is applicable then the trajectory of the particle is governed by the following ordinary differential equations:

$$(dx/dt) = u - K(\partial V/\partial x) \quad , \quad x = x_o \quad \text{at} \quad t = 0$$

$$(dy/dt) = v - K(\partial V/Dy) \quad , \quad y = y_o \quad \text{at} \quad t = 0$$

$$(dz/dt) = w - K(\partial V/\partial z) \quad , \quad z = z_o \quad \text{at} \quad t = 0$$

where (u, v, w) are the components of velocity of the neutral gas, V(x, y, z) is the electric potential, and K is the mobility (the signs have been chosen so that k is positive if the charge on the particle is positive).

The velocity components of the neutral gas may be found by solving the appropriate equations of gas dynamics and the electric potential is found from

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2) = 0$$

with V given on certain boundaries which we shall call electrodes.

The case where the neutral gas is incompressible and irrotational is an especially simple one. In this case

$$(dx/dt) = (\partial\phi/\partial x) - K\,(\partial V/\partial x)$$

$$(dy/dt) = (\partial\phi/\partial y) - K\,(\partial V/\partial y)$$

$$(dz/dt) = (\partial\phi/\partial z) - K\,(\partial V/\partial z)$$

where ϕ also satisfies Laplace's equation

$$(\partial^2\phi/\partial x^2) + (\partial^2\phi/\partial y^2) + (\partial^2\phi/\partial z^2) = 0$$

and the gradient of ϕ is specified on certain surfaces in the flow. The pressure in the neutral gas can be expressed in terms of ϕ since

$$p + \tfrac{1}{2}\,\rho\left[(\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2 + (\partial\phi/\partial z)^2\right] = p_\infty + \tfrac{1}{2}\,\rho\,U_\infty^{\,2}$$

where p_∞ and U_∞ are the pressure and velocity of the gas infinitely far upstream. The pressure influences the trajectory of the charged particle through its effect upon mobility. In fact mobilities are independent of temperature to a great extent, and thus the pressure (or density) is the only hydrodynamic variable affecting the mobility.

Therefore,

$$(dx/dt) = (\partial/\partial x)(\phi - K V) + \left[K' (\partial p/\partial x)\right] V$$

$$(dy/dt) = (\partial/\partial y)(\phi - K V) + \left[K' (\partial p/\partial y)\right] V$$

$$(dz/dt) = (\partial/\partial z)(\phi - K V) + \left[K' (\partial p/\partial z)\right] V$$

where $K' \equiv (dK/dp)$ and p is a given function of ϕ.

If the influence of pressure upon mobility is ignored then

$$(dx/dt) = (\partial/\partial x)(\phi - K V)$$

$$(dy/dt) = (\partial/\partial y)(\phi - K V)$$

$$(dz/dt) = (\partial/\partial z)(\phi - K V)$$

and the charged particle moves perpendicular to the surfaces

$$\phi - K V = \text{constant.}$$

Methods for determining ϕ and V are well known and therefore the problem is reduced to that of finding the orthogonal trajectories of a given family of surfaces.

We observe also that if we introduce units of length, velocity and voltage then the potentials occurring in the last equation can be non-dimensionalized as follows:

$$\phi \sim \phi/U_\infty \ell \quad , \quad V \sim V/V_o$$

and therefore there is only a single non-dimensional dynamical parameter involved in the problem:

$$a = (KV_o/U_\infty \ell) \quad .$$

($[(k \Delta V)/\ell]$ is a measure of the velocity due to the electric field, U_∞ is a measure of the velocity of the neutral gas.)

If we imagine a continuous distribution of charged particles instead of just a single particle, then a velocity field exists at each point of space and the quantity

$$\Phi = \phi - K V$$

is a velocity potential, from which the charged particle velocity field can be derived by differentiation.

2. Two-Dimensional Problems

In two dimensions the orthogonal trajectories of the equipotential curves

$$\Phi = \phi - K V = \text{constant}$$

can be found by a simple and convenient method. Let the curves

$$\Psi = \text{constant}$$

be the required orthogonal trajectories. The normal to an equipotential curve has direction numbers

$$(\partial\Phi/\partial x) \ , \ (\partial\Phi/\partial y)$$

and the orthogonal trajectory through the point (x, y) has direction numbers

$$(\partial\Psi/\partial y) \quad , \quad -(\partial\Psi/\partial x) \quad .$$

Therefore the function Ψ (x, y) must satisfy the conditions

$$(\partial\Psi/\partial y) = \lambda \ (\partial\Phi/\partial x)$$

$$-(\partial\Psi/\partial x) = \lambda \ (\partial\Phi/\partial y) \quad .$$

If Ψ is chosen so that $\lambda = 1$ these are the Cauchy-Riemann equations, which are the necessary and sufficient conditions that Φ and Ψ be real and imaginary parts, respectively, of an analytic function of the complex variable $z = x + i y$.

We can therefore proceed as follows:

(a) Determine the fluid and electric potentials ϕ and V.

(b) Then $\Phi = \phi - K V$ is the velocity potential for charged particles.

(c) Find a function of a complex variable f(z) whose real part is Φ (x, y).

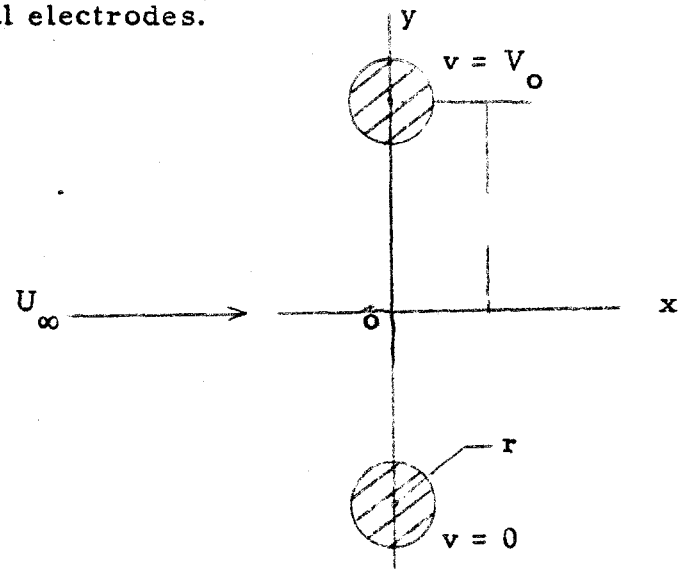
(d) Then if Ψ (x, y) is the imaginary part of f(z), the curves

$$\Psi \ (x, y) = \text{constant}$$

are the paths followed by charged particles.

3. Two Examples

(a) Uniform flow through an electric field produced by a pair of cylindrical electrodes.



If we are not interested in the flow near the electrodes we can ignore the effect of the electrodes upon the flow of the neutral gas, i.e., we do not consider the electrodes as solid boundaries. Then

$$\Phi = U_\infty x - K V_o \left[\tfrac{1}{2} + \frac{1}{2 \log k^2} \ell n \frac{x^2 + (y+\ell)^2}{x^2 + (y-\ell)^2} \right]$$

where $(\ell / \lambda) = (k^2-1)/(k^2+1)$, and $k = \dfrac{1 + \sqrt{1 - (r/\lambda)^2}}{(r/\lambda)}$

$\Phi(x, y)$ is the real part of

$$f(z) = U_\infty z - K V_o \left(\tfrac{1}{2} + \frac{1}{2 \log k} \ell n \frac{z + i\ell}{z - i\ell} \right)$$

and the charged particles follow the curves $\Psi(x, y) = \text{constant}$, where

$$\Psi(x, y) = U_\infty y - \frac{K V_o}{2 \log k} \tan^{-1} \frac{2 \ell x}{x^2 + y^2 - \ell^2} .$$

Introduce dimensionless variables

$$\xi = (x/\ell) \quad , \quad \eta = (y/\ell) \quad , \quad \psi = (\Psi / U_\infty \ell)$$

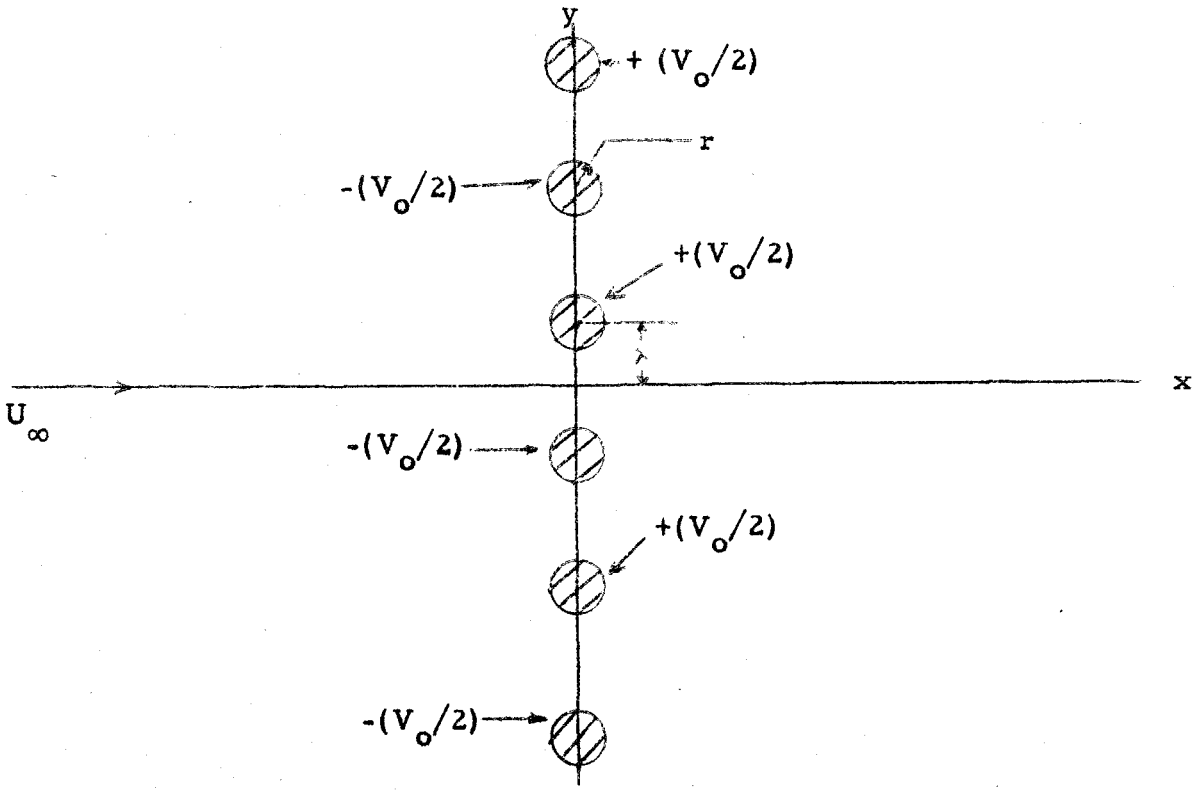
and let $\alpha = (KV_o/\beta U_\infty \ell \log k)$. Then the orthogonal trajectories are found from

$$\psi(x, y) = \eta - \alpha \tan^{-1} \frac{2\xi}{\xi^2 + n^2 - 1} = \text{constant} \quad .$$

Notice that by incorporating the factor $\ell \ln k^2$ into the definition of α and by using ℓ as the unit of length rather than x, we are able to plot one family of trajectories which can be used for any value of (r/λ), that is we have eliminated geometrical parameters and can plot the charged particle paths as functions of a single parameter α which combines both geometrical and dynamical quantities. In general this is not possible. Dividing streamlines and stagnation points are shown in Figure C-1. It is interesting to note that if $\alpha < .55$ then it is not possible for charges to cross the flow from one electrode to the other, i.e., the "arc" between the electrodes is blown out by the flow of neutral gas.

It would be a simple matter to take into account the fact that the electrodes are solid boundaries; however, in this case we would not be able to combine geometrical and dynamical parameters into a single parameter.

(b) Flow through a gridwork of cylindrical electrodes.



A good approximation to the velocity potential ϕ for the neutral gas is obtained by superimposing an infinite array of doublets. Similarly the electric potential V is found by superposing an infinite array of sources and sinks. Therefore,

$$f(z) = \phi - K V$$

$$f(z) = U_\infty z + \sum_{k=-\infty}^{\infty} \frac{\mu}{z - i\lambda(2K+1)}$$

$$- K \left\{ r \sum_{k=-\infty}^{\infty} \log \left[z - i\lambda(4k+1) \right] - r \sum_{k=-\infty}^{\infty} \log \left[z - i\lambda(4k+1) \right] \right\}$$

$$+ \text{arbitrary constant}$$

$$f(z) = U_\infty z + \mu \operatorname{ctnh} \left[(\pi z/2\lambda) - i(\pi/2) \right] + \text{arbitrary constant}$$

$$- K \left\{ r \log \operatorname{ctnh} \left[(\pi z/4\lambda) - i(\pi/4) \right] - r \log \operatorname{ctnh} \left[(\pi z/4\lambda) - i(\pi/4) \right] \right\}$$

(The infinite series are just "partial fraction expansions" of the functions appearing in the last equation.)

The combined potential Φ is the real part of f(z) and the combined stream function Ψ is the imaginary part. The doublet strength μ and the source and sink strength r are related to the radius of the electrodes and the potential difference between them as follows:

$$\mu = (\pi/2)(r/\lambda)^2 (U_\infty \lambda) \quad , \quad r = - \frac{V_o}{4 \log (\pi r/4\lambda)}$$

and the arbitrary constant is chosen so that the potential on the surface of the electrode with center at $x = 0$, $y = \lambda$ is $(V_o/2)$.

$$\text{arbitrary constant} = - \pi K r \quad .$$

In this way we obtain

$$\Psi(x, y) = U_\infty y + (\pi/2)(R/\lambda)^2 U_\infty \lambda \frac{\sin(\pi y/\lambda)}{\cosh(\pi x/\lambda) + \cos(\pi y/\lambda)}$$

$$+ \frac{K V_o}{2 \log(\pi r/4\lambda)} \left(\tan^{-1} \frac{\cos(\pi y/2\lambda)}{\sinh(\pi x/2\lambda)} - \pi\right) .$$

In terms of non-dimensional quantities

$$\xi = (x/\lambda) \quad , \quad \eta = (y/\lambda) \quad , \quad \psi = (\Psi/U_\infty \lambda)$$

$$\boxed{\psi = \eta + (\pi/2)(r/\lambda)^2 \frac{\sin \pi \eta}{\cosh \pi \xi + \cos \pi \eta} - \alpha \left(\tan^{-1} \frac{\cos(\pi \eta/2)}{\sin(\pi \xi/2)} - \pi\right)}$$

where

$$\alpha = \frac{K V_o}{2 U_\infty \lambda \log(4\lambda/\pi r)} .$$

Streamlines are plotted in Figures C- 2, 3, and 4 for $\alpha = .25, .5$, and 1.0 with $(r/\lambda) = .5$.

It should be pointed out that the solution we have presented is only valid for small values of (r/λ). If large values of (r/λ) are used in our formulas then the circular electrodes will not be streamlines, even approximately. Also the voltage on the surface of the electrodes will not be constant. However, even for $(r/\lambda) = .5$ the approximation is seen to be rather good, in the sense that the position of the electrodes, calculated from the stream function for the neutral gas, almost gives circles.

If we fasten our attention on that part of the flow included between $y = -\lambda$ and $y = +\lambda$ then we have the solution to the problem of flow in a duct with semicircular electrodes.

4. Distribution of Charged Particles

If we imagine that charged particles are distributed throughout the flow field then we may wish to know the density n of charged particles at each point of the flow. If there is no production of charged particles in the flow then

$$(\partial/\partial x)(nU) + (\partial/\partial y)(nV) + (\partial/\partial z)(nW) = 0$$

where (U, V, W) is the velocity vector of the charged particle fluid at the point (x, y, z).

$$U = u - K(\partial V/\partial x)$$

$$V = v - K(\partial V/\partial y)$$

$$W = w - K(\partial V/\partial z)$$

where (u, v, w) is the velocity vector of the neutral fluid.

Combining these four equations gives

$$U(\partial n/\partial x) + V(\partial n/\partial y) + W(\partial n/\partial z) = -n\left[(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z)\right]$$

$$+ nK'\left[(\partial p/\partial x)(\partial V/\partial x) + (\partial p/\partial y)(\partial V/\partial y) + (\partial p/\partial z)(\partial V/\partial z)\right]$$

$$+ nK\left[(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2)\right] .$$

If the right-hand side of this equation were zero then the equation would simply be equivalent to the statement that the number density is constant along the streamlines of the charged particles. This would be the case if

(a) The neutral gas is incompressible, so that
$(\partial u/\partial x) + (\partial v/\partial y) + (\partial w/\partial z) = 0$.

(b) The effect of pressure upon mobility is negligible.

(c) The electric potential is determined only by externally applied voltages and is not at all influenced by the distributed charges in the flow. Under such circumstances

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2) = 0 \quad .$$

We have already seen that if all three of these conditions are satisfied, there exists a velocity potential Φ for the velocity field of the charged particles, i.e.,

$$U = (\partial\Phi/\partial x) \quad , \quad V = (\partial\Phi/\partial y) \quad , \quad W = (\partial\Phi/\partial z) \quad .$$

In this case, the streamlines are just the orthogonal trajectories of the family of surfaces

$$\Phi = \text{constant}$$

and n is constant along a streamline, although n can assume different values along different streamlines.

In general, this simple situation does not occur, and the calculation of n is complicated by the fact that the streamlines of the charged particles cannot be calculated independently of the charge density, since

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2) = - r n \quad .$$

The two equations

$$(\partial/\partial x)\ n\left\{\left[u - K\,(\partial V/\partial x)\right]\right\} + (\partial/\partial y)\left\{n\left[v - K\,(\partial V/\partial y)\right]\right\} + (\partial/\partial z)\left\{n\left[w - K(\partial V/\partial z)\right]\right\} = 0$$

$$(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) + (\partial^2 V/\partial z^2) = - r n$$

have to be solved simultaneously, with u, v, w, K given as prescribed functions of x, y, z (K depends upon x, y, z through its dependence on the pressure p of the neutral gas). It might be possible to solve these equations by means of an iteration procedure, in which one makes an initial guess for n(x, y, z), then computes the corresponding V(x, y, z) from Poisson's equation. This potential distribution can then be used to construct a new estimate of n(x, y, z), etc.

Numerical calculations for the figures in this appendix were performed by Mrs. Truus van Harreveld.

FIGURE C-1

DIVIDING STREAMLINES FOR SEVERAL VALUES OF α

Anode surface

$$\left(\frac{x}{L}\right)^2 + \left(\frac{y}{L} + \frac{\lambda}{L}\right)^2 = \left(\frac{r}{L}\right)^2$$

Cathode surface

$$\left(\frac{x}{L}\right)^2 + \left(\frac{y}{L} + \frac{\lambda}{L}\right)^2 = \left(\frac{r}{L}\right)^2$$

V volts

r

λ

U cm/sec

K = ion mobility (cm/sec / volt/cm)

o volts

$$\alpha = \frac{KV}{U \ln k^2}$$

$$\frac{\lambda}{r} = \frac{k^2-1}{k^2+1} \qquad k = \frac{1 + (1 - r^2/\lambda^2)^{1/2}}{r/\lambda}$$

$\alpha = 2$

$\alpha = 1$

$\alpha = .55$

$\alpha = .5$

$\alpha = .2$

$\alpha = .1$

$\alpha = 0$

$\alpha = -.1$

$\alpha = -.2$

$\alpha = -.5$

$\alpha = -1$

$\alpha = -2$

x/L

y/x

$\alpha = .25$, $r/\lambda = .5$

GAGE CL

CHARGED PARTICLE SURFACE

η

$\eta=0$ $\zeta=0$ $\xi=0$

$\gamma = 1.00$ $\frac{\gamma}{2} = 1.785$

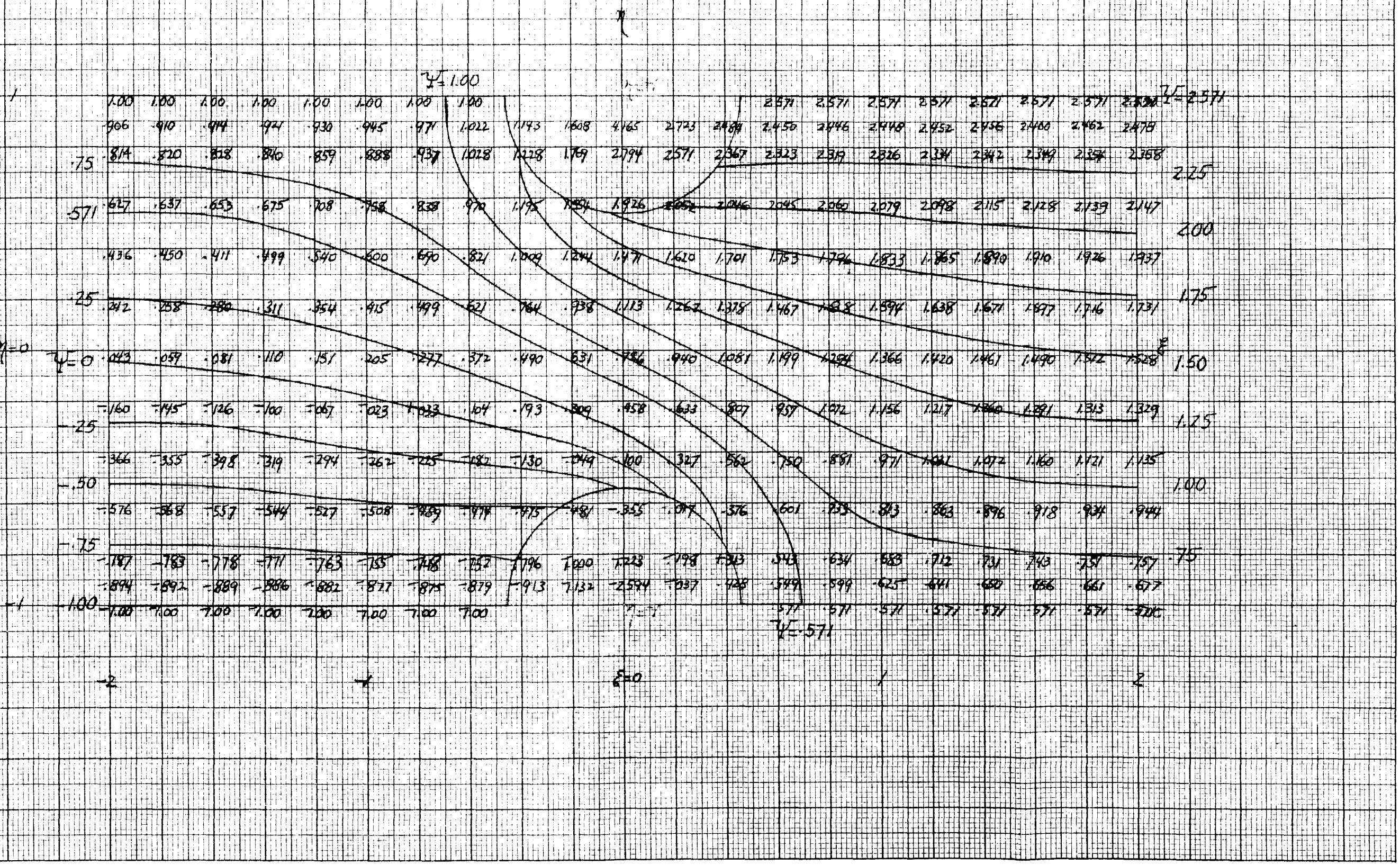
$\gamma = .215$

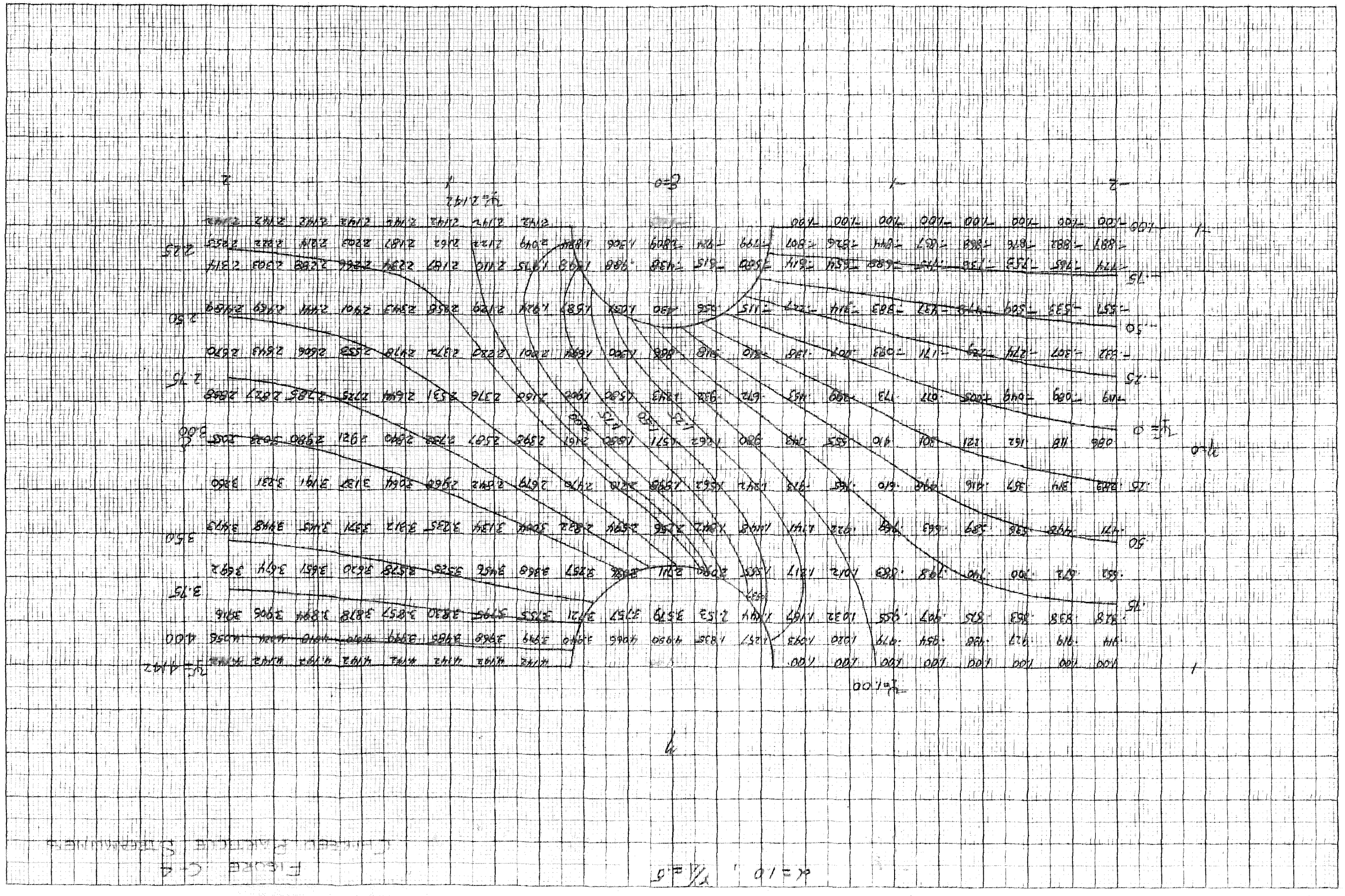
η \ ξ	-2	-1	0	1	2
.75	.808 .811 .816 .823 .835 .855 .890 .950 1.120 1.577 2.401 1.978 1.689 1.606 1.581 1.573 1.572 1.578 1.576 1.578 1.580				
.50	.644 .620 .628 .642 .663 .696 .751 .846 1.015 1.286 1.533 1.534 1.440 1.384 1.361 1.358 1.362 1.366 1.371 1.374				
.25	.419 .427 .438 .454 .479 .516 .573 .661 .789 .946 1.078 1.141 1.135 1.127 1.127 1.132 1.141 1.150 1.157 1.164 1.169				
0	.221 .230 .241 .258 .282 .317 .366 .435 .524 .626 .720 .788 .831 .861 .885 .906 .924 .938 .950 .959 .966				
-.25	.082 .090 .100 .115 .135 .162 .202 .259 .339 .439 .540 .620 .677 .717 .747 .770 .788 .802 .813 .822 .829				
-.50	-.075 -.066 -.054 -.038 -.018 .008 .045 .098 .175 .275 .380 .464 .523 .566 .599 .625 .645 .661 .674 .685 .693				
-.75	-.215 -.204 -.190 -.172 -.150 -.121 -.080 -.026 .046 .138 .253 .357 .440 .503 .551 .589 .619 .643 .662 .677 .689				
-1.00	-.384 -.379 -.372 -.364 -.355 -.347 -.341 -.342 -.349 -.348 -.293 -.160 .008 .124 .212 .269 .307 .334 .348 .359 .367				
-1.25	-.589 -.585 -.581 -.576 -.572 -.571 -.576 -.598 -.655 -.749 -.748 -.500 -.239 -.060 .035 .089 .123 .143 .157 .165 .171				
-1.50	-.794 -.792 -.790 -.788 -.787 -.788 -.795 -.821 -.904 -1.192 -1.616 -.791 -.334 -.174 -.105 -.069 -.050 -.038 -.030 -.025 -.022				
-1.75	-.897 -.896 -.895 -.894 -.894 -.894 -.899 -.914 -.971 -1.266 -1.987 -.708 -.310 -.200 -.162 -.143 -.133 -.126 -.123 -.120 -.118				
-2.00	-1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00 -1.00				

$\alpha = .5,\ r/\lambda = .5$

FIGURE C-3

CHANGED PARTIAL STRESS FIELD





APPENDIX D

TRANSIENT MOTION OF SPACE-CHARGE IN A PULSE-IRRADIATED GAP

The chemically frozen motion of charged particles in a cold, neutral gas, forms the substance of the so-called space-charge problem in gaseous electronics. One customary approach to the solution of this problem is to assume near-neutrality of the mixture and thus to linearize the perturbation induced in the applied electric field strength. Variants of the non-linear problem have also been solved with the aid of computers, but often in ways detrimental to the understanding of the problem and to the generality of the results.

The present paper presents an exact solution to the transient problem of charged particles moving between two plane parallel electrodes, in a manner completely dominated by the local electric field strength, and including an arbitrarily large effect of the induced component of the field. In order to illuminate the effect of the induced field, pure diffusion and chemical reactions, such as ionization, are not included in the analysis. Since these effects are in many cases negligible, however, it is expected that direct comparison of the present results with experiments should often be possible.

To fix ideas, we will consider that the gas between two plane parallel electrodes is irradiated at some instant so that a certain given number of ion-electron pairs are created; immediately afterwards the

particles begin drifting to their respective electrodes under the action of an impressed potential difference across the gap. It is not unreasonable to expect that the electrons, being extremely mobile, will be depleted long before the ions have transversed any appreciable distance in the gap. Consequently, we will seek exact solution to two distinct problems: (a) the motion of the electrons in a stationary ion distribution, and (b) the subsequent motion of the ions in the gap already depleted of electrons.

Choice of an Analytical Model

The two conduction processes enunciated are taking place under conditions such as shown in Figure 1. The inter-electrode distance d is so small compared to the transverse dimension of the electrodes, that the particle motion is chiefly along x, the coordinate normal to the electrode surface. An external source maintains a potential difference V_0 across the electrodes such that

$$\int_{x=0}^{x=d} E\,dx = V_0 \tag{1}$$

where E is the local electric field. Further, the irradiation process fills the gap with uniformly distributed charges n^+ and n^- cm^{-3} so that

$$n^+(t=0) = n^-(t=0) \equiv n_0 \text{ , const.} \tag{2}$$

In the absence of reactions which affect the number density of each specie, the continuity equations for the species are:

$$\frac{\partial n^{\pm}}{\partial t} + \frac{\partial}{\partial x} n^{\pm} v^{\pm} = 0 \tag{3}$$

where $v^{\pm}$ is the velocity of the specie; a positive velocity indicates motion of the particle in question towards the cathode, as seen in Figure 1.

The equation of motion of the particles of each specie obey

$$v^{\pm} = \pm K^{\pm} E \tag{4}$$

where $K^{\pm}$ are positive scalars equal to the mobility of the species. This expression[1] must be used because of convenience, and with due regard to the restrictions which bound its range of validity. In addition to neglecting the inertia of the particles, Eq. (4) actually states that the local electric fields are at all times small enough so that the $K^{\pm}$ are not themselves functions of the field strength.[2] It furthermore states that E is strong enough so that the charged-particle motion is controlled by the field rather than pure diffusion of the species. These two bounds on the field strength can generally be checked a posteriori, once the solution yields the field strength at each point in the gap.

The Poisson equation

$$\frac{\partial E}{\partial x} = 4\pi q_f (n^+ - n^-) \tag{5}$$

where $q_f \equiv q_f^+ = q_f^-$ completes (together with (3) and (4)) the system of five equations needed to solve for $n^{\pm}$, $v^{\pm}$ and E.

---

1. Loeb, L., Basic Processes of Gaseous Electronics, University of California Press, Berkeley, 1955.

2. Demetriades, A., Electric-Field Heating Threshold for Charged Particles, Physics of Fluids, 5, No. 8, p 1134, Sept. 1962.

Solution for the Electron Motion

As indicated above the electrons should, by virtue of their superior mobility,[1] be able to move across the gap under the action of the applied field $E_0 \equiv \frac{V_0}{d}$ and within an interval during which the ions can be considered immobile. The electron-motion model will therefore involve the legitimate approximation

$$K^+ = 0$$

$$n^+ = n_0 \tag{6}$$

whereas $\bar{K} \equiv K$ is a finite constant encompassing pertinent parameters such as the density of the cold gas, its temperature, etc. It is to be expected that the polarization of the gap resulting from the electron motion may on occasion overcome the applied field strength, as will be seen below.

The five equations (3) - (5) are reduced to three by means of (6). It will now be advantageous to non-dimensionalize the various quantities by using the known field strength E_0, gap separation d, and the electron mobility $\bar{K}$ as follows:

$$\frac{x}{d} \rightarrow x \qquad \frac{E}{E_0} \rightarrow E$$

$$\frac{\bar{K} E_0 t}{d} \rightarrow t \qquad \frac{V_0}{d E_0} \rightarrow l$$

$$\frac{n^-}{n_0} \rightarrow n \tag{7}$$

$$\frac{v^-}{\bar{K} E_0} \rightarrow v \qquad \frac{4\pi q n_0 d}{E_0} \rightarrow s$$

In these expressions the l.h.s. symbols are always dimensional physical quantities, whereas those on the r.h.s. are dimensionless. The equations describing the electron motion for $t>0$ assume the non-dimensional form:

$$\frac{\partial E}{\partial x} = S(1-n) \tag{8}$$

$$\frac{\partial n}{\partial t} - \frac{\partial}{\partial x} nE = 0 \tag{9}$$

and the boundary conditions become:

$$n(t=0) = 1 \tag{10}$$

$$\int_0^1 E\,dx = 1 \tag{11}$$

The sole parameter of the problem is S, which is a measure of an induced to the applied electric field as indicated by (7). For $S=0$ the problem becomes the trivial one of the wave-like advance of the initial electron distribution towards the anode. This simple consideration, extended to the case where the induced field is appreciable compared to the applied field ($S \neq 0$) is, in fact, the key to the entire solution; for it is obvious that since no diffusive mechansim appears in (4), a discontinuity in the electron distribution should exist in the gap once their motion begins.*

*These discontinuities are the familiar "characteristics" exhibited by hyperbolic problems such as the one at hand.

In particular, we expect that this discontinuity or "front" consisting of an electron "rearguard" and located at some $x = \xi$ such that $0 < \xi < 1$ separates the gap into two regions: electrons exist in $0 < x < \xi$ but not in $\xi < x < 1$. In this latter regions the depleted electrons have left exposed a uniform and stationary ion distribution $n^+ = n_0$ (cf. e.g., (6)), and hence the solution is here known, within some constant factors. Matching this solution with that obtained in the region $0 < x < \xi$ from (8) and (9) will complete the problem.

To solve the system (8), (9), in $0 < x < \xi$ we combine the two equations into

$$E \frac{\partial^2 E}{\partial x^2} + \left(\frac{\partial E}{\partial x} \right)^2 - s \frac{\partial E}{\partial x} - \frac{\partial^2 E}{\partial x \partial t} = 0 \qquad (12)$$

This equation is susceptible to the solution by the so-called Monge method [3] which is found to be of the form

$$\frac{\partial E}{\partial x} \sim \frac{s}{1 - e^{s(t+c)}}$$

To satisfy (10), however, $c \to \infty$ and consequently

$$E = T(t) \qquad 0 < x < \xi \qquad (13)$$

---

3. Miller, F. H., Partial Differential Equations, John Wiley & Sons, New York, 1941.

That is, the field strength is constant between the anode and the front, but its level varies with time. The electron density in the same region is immediately obtained from (8):

$$n = 1 \tag{14}$$

so that the electrons move bodily (but with a time-dependent velocity) into the anode.

The solution for E in the region $0 < x < \xi$ cannot be completed till $T(t)$ is found (cf. (13)). This will now be accomplished by first considering the solution in $\xi(t) < x < 1$ and the motion of the front ξ.

In the region $\xi < x < 1$, $n = 0$ and from (8) we obtain immediately

$$E(x,t) = sx - s\xi + T(t) \qquad \xi < x < 1 \tag{15}$$

Now, by the condition (11),

$$\int_0^1 E\,dx = \int_0^{\xi} E\,dx + \int_{\xi}^1 E\,dx = 1 \tag{16}$$

But between 0 and ξ, E is given by (13), whereas between ξ and 1 it is given by (15). Making these substitutions we obtain

$$T(t) = -\frac{s\xi^2}{2} + s\xi + \left(1 - \frac{s}{2}\right) \tag{17}$$

However, the front velocity $d\xi_f/dt$ towards the anode is identical with the particle velocity at ξ_f ; and in our non-dimensional notation this means

$$\frac{d\xi_f}{dt} = -T(t) \tag{18}$$

whence we get the equation of the front motion

$$\frac{d\xi_f}{dt} = \frac{s\xi_f^2}{2} - s\xi_f + \left(\frac{s}{2} - 1\right) \tag{19}$$

with the result

$$\xi_f = \left[1 - \sqrt{\frac{2}{s}} \tanh\left(\sqrt{\frac{s}{2}}\, t\right)\right] \tag{20}$$

and the solution is complete. To recapitulate, a "front" located according to (20) appears after $t=0$. At any time t later the field strength in the region $0<x<\xi_f$ is given by (17), with ξ_f as given by (20); both electrons and ions retain their original uniform number density in this region. In the remainder of the gap, $\xi_f<x<1$, the field is given by (15), with $\xi_f^{T(t)}$ from (17) and ξ_f from (20). In this region there are no electrons ($n=0$) whereas the ions retain their original distribution.

Figure 2 shows a time sequence in the motion of the front and the variation of the electric field in the gap. We can define a non-dimensional "depletion" time t_d as the time required for the gap to become depleted of electrons, i.e., for ξ to become zero:

$$t_d = \frac{1}{\sqrt{2S}} \ln \left[\frac{S + \sqrt{2S}}{\sqrt{2S} - S} \right] \tag{21}$$

Note that

$$\lim_{S \to 0} t_d = 1$$

which, on the basis of our non-dimensionalization (cf. Eqs. (7)) means that the electrons will evacuate the gap in a time $d/\kappa E_0$ as expected, when there is no space-charge effect.

The case pictured in Figure 2 is labeled "intermediate" because $S < 2$ here; at $S = 2$ the front needs infinite time to arrive at the anode, according to (21). To appreciate the significance of this latter situation, consider the case pictured in Figure 3, where $S > 2$. Here the field strength in $0 < x < \xi$ decreases so fast that it becomes zero before the front arrives at the anode. The front therefore comes to a complete stop at a distance ξ_s from the anode (cf. Figure 3(c)) such that

$$\xi_s = 1 - \frac{\sqrt{2S}}{S} \tag{22}$$

where it remains from then on. Obviously, for very high initial charge densities, $S \to \infty$ and the electrons can only move a minute distance towards the anode before they come to a stop.

The value $S = 2$ thus marks a threshold above which the space-charge effects are truly "strong"; for any given applied field E_0 and inter-electrode distance d, the initial electron density n_0 which will give rise to this situation can be computed directly from (7). Clearly, to complete the study of electron motion, one should examine the events subsequent to the electrons coming to a stop, a problem which involves the simultaneous motion of both electrons and ions. The usefulness of the chosen model, wherein the electrons move in a stationary distribution of ions is therefore restricted to $S < 2$. The case $S > 2$ will not be examined.

Solution of the Ion Motion

In the previous section we found exact solutions for the electron motion, although those solutions with $S > 2$ cannot be used to describe all events before the electrons are depleted and the ions begin moving. In the present section we will solve as a sequel, the problem of the ions themselves moving once all the electrons have been depleted.

The formulation of this problem is practically identical to that of the previous section. Now, however,

$$n^- \equiv 0$$

$$\frac{n^+}{n_0} \equiv n$$

$$K^+ \equiv K$$

where n_0 is the initial ion distribution. The pertinent equations are, again in non-dimensional form (cf. (7)),

$$\frac{\partial E}{\partial x} = Sn \tag{23}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} nE = 0 \tag{24}$$

and the boundary conditions,

$$n\,(t=0) = 1 \tag{25}$$

$$\int_0^1 E\,dx = 1 \tag{26}$$

again. Note that the positive sign in (24) accounts for the fact that the ions move towards the cathode and hence have a positive velocity. Also note that whereas the initial field strength was constant in the previous problem, the field is now initially a linear function of x, as seen from (23) and (25).

The solution is again straightforward once the existence of a front is recognized. The "rearguard" of the ions advancing towards the cathode is located at ξ so that the region $0 < x < \xi$ is obviously devoid of all charged particles:

$$n = 0 \qquad 0 < x < \xi \tag{27}$$

and therefore

$$E = P(t) \qquad 0 < x < \xi \tag{28}$$

follows from (23). In the region $\xi < x < 1$ we have to solve (23) and (24) simultaneously. When combined these give

$$\frac{\partial^2 E}{\partial x \partial t} + \frac{\partial}{\partial x} E \frac{\partial E}{\partial x} = 0 \tag{29}$$

Again applying Monge's method and condition (25) we obtain

$$E(x,t) = \frac{sx}{st+1} + Q(t) \qquad \xi < x < 1 \tag{30}$$

which, at $x = \xi$ must equal (28), that is:

$$P(t) = \frac{s\xi}{st+1} + Q(t) \qquad @ \; x = \xi \tag{31}$$

Since P and Q are unknown we need a second equation relating them; this is supplied by the condition (26). It can be easily shown that this condition is correctly written

$$\int_0^1 E\,dx = \int_0^{\xi} P(t)\,dx + \int_{\xi}^1 \left[\frac{sx}{st+1} + Q\right] dx = 1 \qquad s < 2 \tag{32}$$

$$\int_0^1 E\,dx = \int_0^1 \left[\frac{sx}{st+1} + Q\right] dx = 1 \qquad s > 2 \tag{33}$$

Let us first consider (32). Integration gives, together with (31),

$$Q(t) = 1 - \frac{s}{st+1}\left(\frac{1}{2} + \frac{\xi^2}{2}\right) \tag{34}$$

and $P(t)$ follows at once from (31). As in the previous case of the electron motion, the problem reduces to finding the front position as a function of the time. The differential equation of the front motion is again obtained by realizing that the front and ion velocities are identical at the location of the front:

$$\frac{d\xi}{dt} = P(t)$$

whence

$$\frac{d\xi}{dt} = 1 - \frac{s(\xi-1)^2}{2(st+1)} \tag{35}$$

This relation has the form of Ricatti's equation and can further be reduced to Bessel's equation by an appropriate transformation. The result* is

$$\xi(t) = 1 + \sqrt{2z}\,\frac{I_1(\sqrt{2z}) - \delta K_1(\sqrt{2z})}{I_0(\sqrt{2z}) + \delta K_0(\sqrt{2z})} \tag{36}$$

---

* This solution has also been obtained by J. A. Morrison and D. Edelson (Journal of Applied Physics, Volume 33, page 1714-1720, May 1962).

where

$$Z \equiv t + \frac{1}{s}$$

and I_0 etc., are the modified Bessel functions; the constant δ is

$$\delta \equiv \frac{I_0\left(\sqrt{\frac{2}{s}}\right) + \sqrt{\frac{2}{s}}\, I_1\left(\sqrt{\frac{2}{s}}\right)}{-K_0\left(\sqrt{\frac{2}{s}}\right) + \sqrt{\frac{2}{s}}\, K_1\left(\sqrt{\frac{2}{s}}\right)} \tag{37}$$

The electric field strength $P(t)$ in the region $0 < x < \xi$ is obtainable from (31), (34), and (36); the charge density there is zero (cf. Eq. (27)). In the range $\xi < x < 1$ the solution is obtained directly from (30) with the aid of (34) and (36). The charge density n in the same region is given by (23) and (30) and is

$$n(t) = \frac{1}{st+1} \qquad \xi < x < 1 \tag{38}$$

The sequence of events is shown on Figure 4. Here the front appears at the anode and moves towards the cathode (since we are considering ions) at a rate given by (36). At $t = 0$ the field has a constant gradient while at the end it is constant at the value E_0.

Figure 4 describes the events for $s < 2$, that is, when the initial ion density has a small-to-intermediate value. Since s is numerically equal to the field gradient at $t = 0$, the above statement shows that initially the field (and hence also the ion velocity) at the anode is positive, as shown

in Figure 4(a); therefore the ions there move towards the cathode right from the start. When, however, $s > 2$ then the initial field at the anode is negative, and ions actually move towards the anode. Simply stated, when the initial ion density is very large, the particles travel outward towards both electrodes due to self-repulsion irrespective of the externally-imposed polarity.

When $s > 2$, the solution obtained in Eqs. (34), etc., far from being invalid, comprises the second of two distinctly different but continuous phases of the ion conduction process. The first phase consists of a gradual loss of the ions (both to the anode and the cathode) till the slope of the field again becomes 2, at which time the solution as described by Equations (28), (30), (34), (36), etc., takes over.

To solve for this initial phase of the case $s > 2$, we use equations (31) and (33). The result is, simply,

$$Q(t) = 1 - \frac{1}{2\left(t + \frac{1}{s}\right)} \tag{39}$$

and

$$E(x,t) = 1 + \frac{s}{st+1}\left(x - \frac{1}{2}\right) \qquad \text{all } x \tag{40}$$

while

$$n = \frac{1}{st+1} \qquad \text{all } x \tag{41}$$

This phase of the ion motion is characterized by an absence of a front; it is easily seen that the front will appear within a time

$$t' = \frac{1}{2} - \frac{1}{S}$$

at which time the slope of the field is 2. Therefore, the remainder of the solution for later times is now obtained by (28) through (38) with $S = 2$.

These events are illustrated in Figure 5, with the front appearing in Figure 5(b). It should also be noted that in this case of ion motion, time is non-dimensionalized with the ion rather than the electron mobility (cf. (7)).

Summary and Conclusions

This paper has presented exact solutions to two distinct space-charge problems.

In the first, a specie of negatively charged particles of initially uniform number density n_0 between two electrodes advances against a stationary background of positive particles of uniform density n_0 . A "front" appears moving according to (20) and across which the negative particle density jumps from zero (for $\frac{2}{3} < x < 1$) to its initial value (for $0 < x < \frac{2}{3}$). The field strength is given by (15) in the former region, and by (17) in the latter. When the non-dimensional parameter S , which compares the initial induced to the externally applied field, is smaller than 2, the solution is useful to the point where all moving particles are depleted from the gap. When $S > 2$ some of the negative particles are "trapped" in the gap and come to a stop when the induced field just balances the applied field. No solution is available for later times in this case.

In the second problem, positive ions are the only charged specie contained in the gap; the problem, of course, is generalizeable to any single-sign specie. Exact solutions are found for the conduction process, which is shown to occur in one or two phases corresponding to $S < 2$ and $S > 2$. In the latter case the solution for the field strength is given by (40) for $0 < t < \frac{1}{2} - \frac{1}{S}$ and by equations (28), (30), (34), and (36) with $S = 2$ for later times. For $S < 2$ the solution is again obtained from the latter four equations for whatever S characterizes the problem.

It is suggested that this initially negative and subsequently positive conduction process resembles the electron and ion motion in a suddenly-irradiated air gap between two plane parallel electrodes.

Acknowledgements

The work described in the present paper was supported by MHD Research, Inc. under Contract No. 29(601)5202 with the U. S. Air Force, Special Weapons Center.

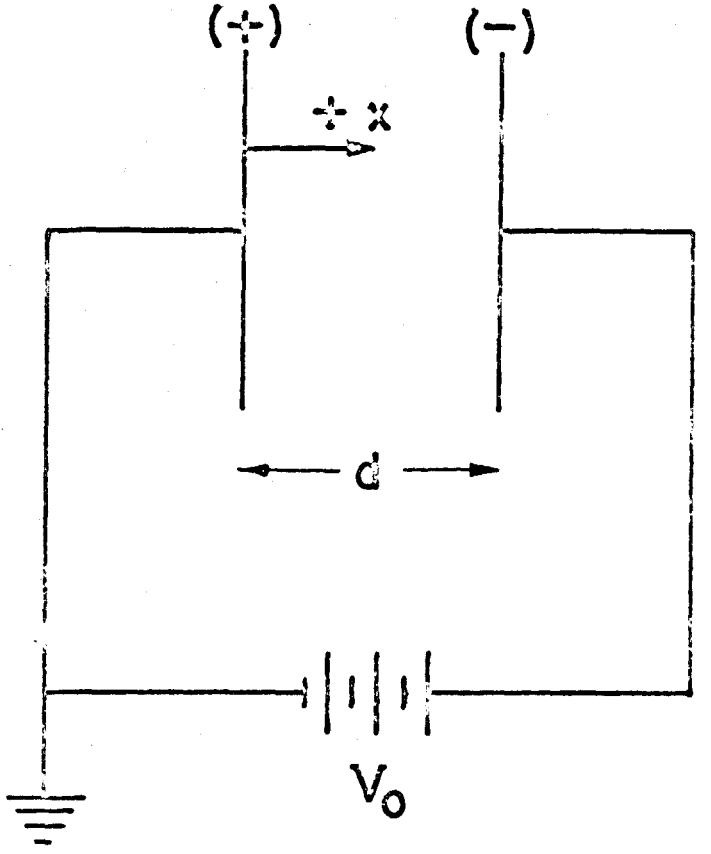


FIGURE 1

Schematic diagram for the conduction model. Note the independence of anode and cathode currents.

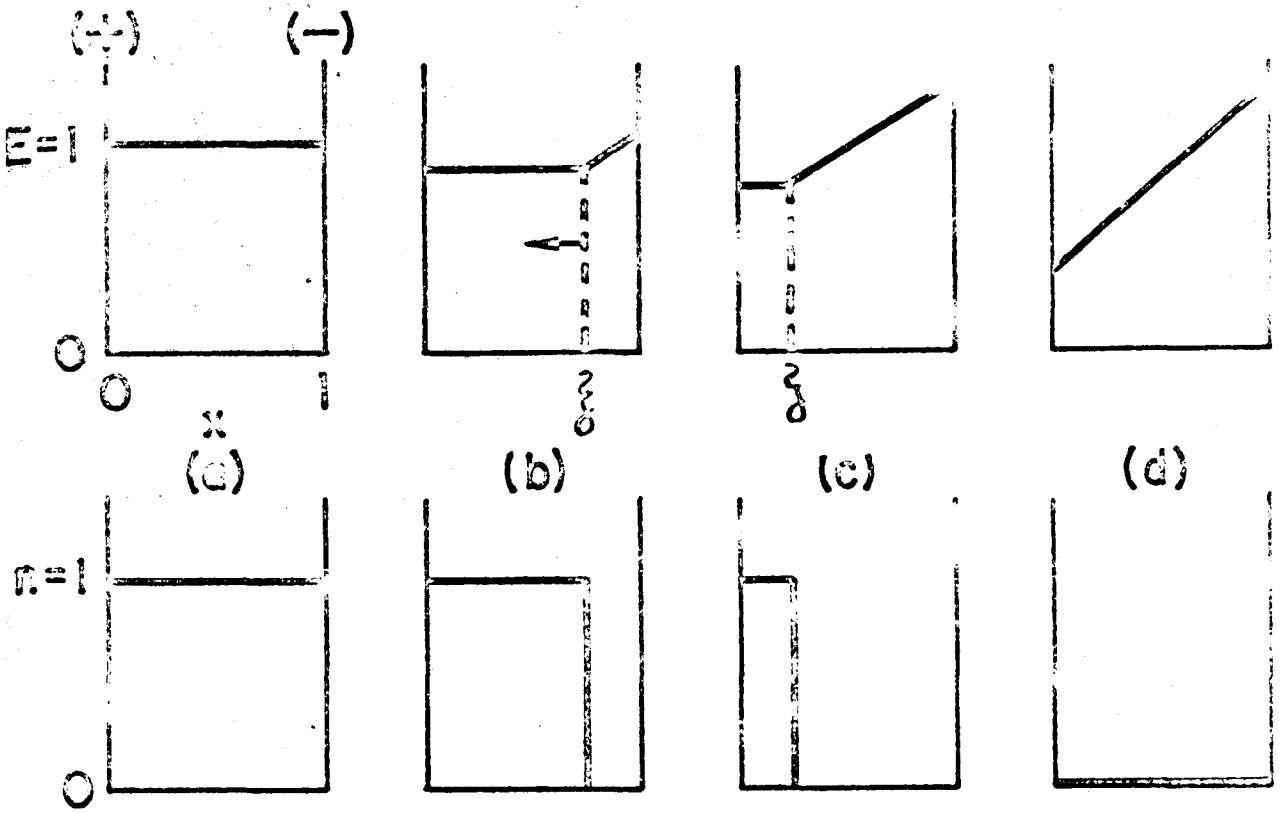


FIGURE 2

Qualitative time sequence of the variation of the field (top) and electron density (below) for "intermediate" space charge ($0 < \alpha < 2$). Sketch (a) corresponds to t = o; in (d) the electrons are depleted and the resulting field gradient is due to the ions.

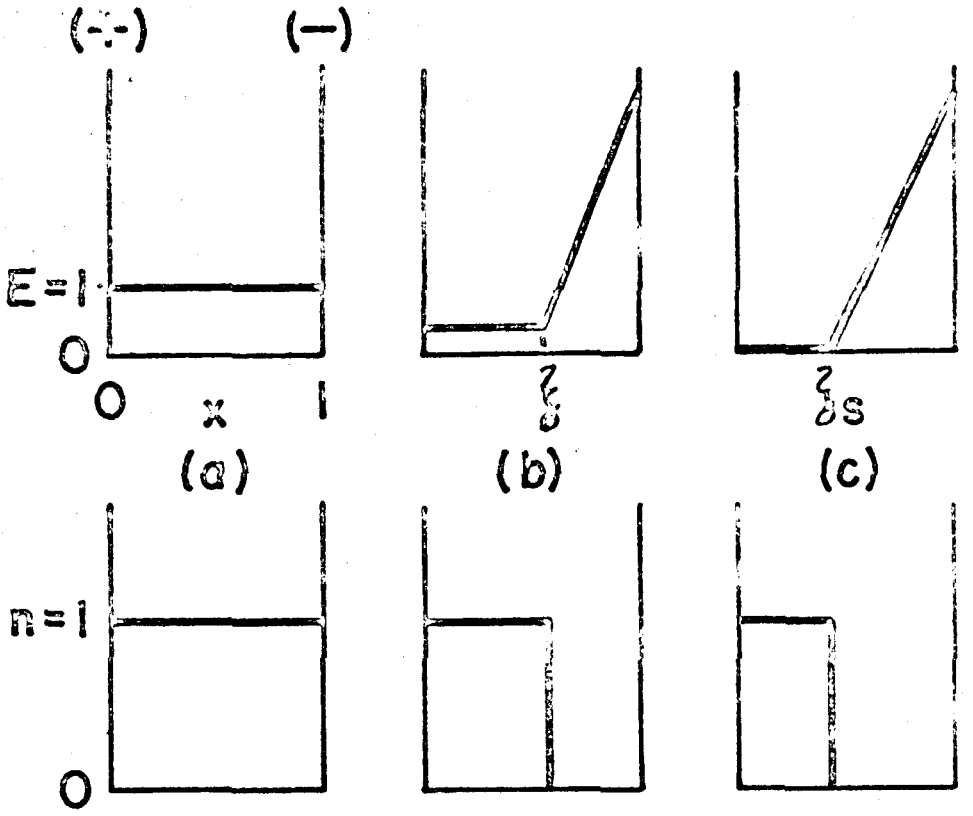


FIGURE 3

Qualitative variation of the field and electron density for "strong" space charge ($\delta > 2$). No further motion is possible once the situation pictured at (c) is reached.

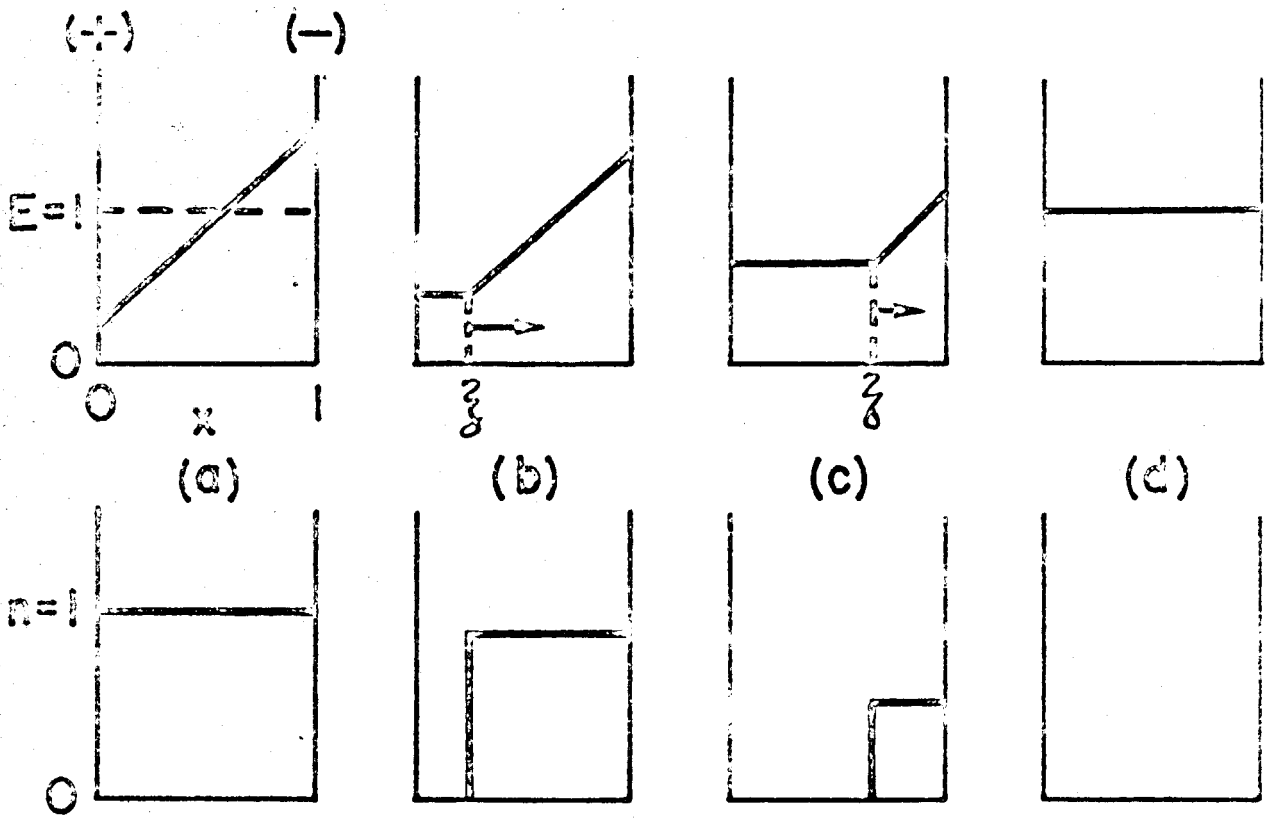


FIGURE 4

Qualitative time sequence of the variation of the field (top) and ion density (below) for intermediate space charge ($0 < \alpha < 2$). Sketch (a) shows the situation at $t = o$.

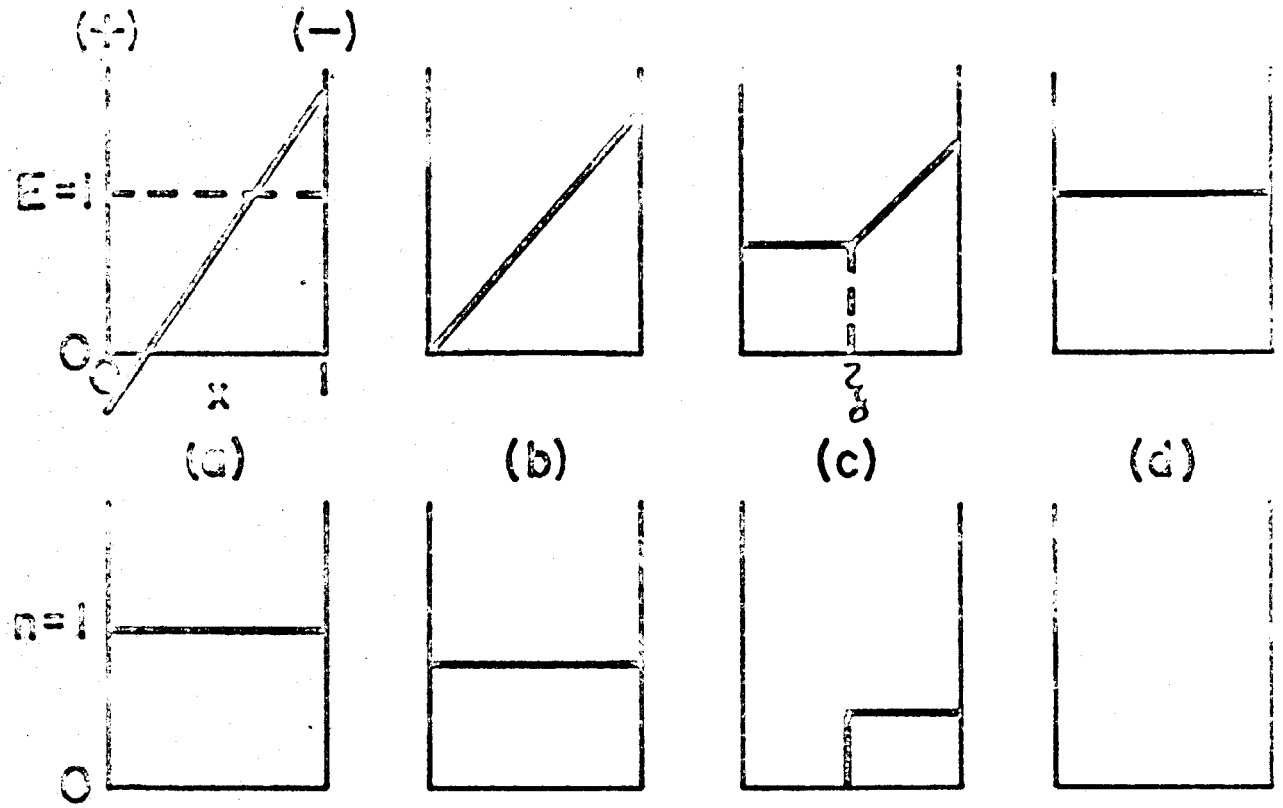


FIGURE 5

Qualitative variation of the field and ion density for strong space charge ($\alpha > 2$). Beginning with (b) the solution is identical with that for $0 < \alpha < 2$.

APPENDIX E

ONE DIMENSIONAL TRANSIENT MOTION OF CHARGED PARTICLES

1. Equations of Motion and a Method of Solution

Imagine a pair of planar electrodes separated by a gap of width d (as in Figure E-1). Assume that the gap contains a mixture of three components:

(a) a neutral gas

(b) positively charged particles (ions)

(c) negatively charged particles (electrons or atoms with an attached electron).

Furthermore, we shall assume that the neutral gas is stationary, and that the charged particles move through the neutral gas according to a simple mobility law, i. e., the velocity of a charged particle is proportional to the local electric field intensity at the location of the particle, and the constant coefficient of proportionality is called the mobility of the particle. Recombination and ionization processes will be ignored; thus charged particles are not being created or destroyed except at the electrodes. A constant electric potential difference ΔV is assumed to exist across the gap.

Since diffusion and inertia of the charged particles are ignored, the non-dimensional equations of motion can be written

$$(\partial n/\partial t) + (\partial/\partial x)(nE) = 0 \qquad \text{(conservation of negative particles)} \qquad \text{(E-1)}$$

$$(\partial N/\partial t) + (\partial/\partial x)(N\alpha E) = 0 \qquad \text{(conservation of positive particles)} \qquad \text{(E-2)}$$

$$(\partial E/\partial x) = s(n - N) \qquad \text{(Poisson's equation)} \qquad \text{(E-3)}$$

where the non-dimensional variables have the following meanings:

n = negative particle number density, in units of some basic number density n_o

N = positive particle number density, same units as n

E = voltage gradient in units of $(\Delta V/d)$

x = length in units of d

t = time in units of $(d^2/k\Delta V)$, where k is the mobility of the negative particles.

Note that $(d^2/ k \Delta V)$ is the transit time of the negative particles when the induced field can be ignored ($s = 0$). Only two parameters appear in the equations of motion; these are

$$\alpha = \frac{\text{positive particle mobility}}{\text{negative particle mobility}}$$

$$s = \frac{e\, n_o\, d^2}{\epsilon\, \Delta V}$$

where α is negative since the mobilities of positive and negative particles are of opposite sign, and where

e = electronic charge

ϵ = dielectric constant of the neutral fas filling the space between the electrodes

The parameter s measures the importance of the electric field produced by the charged particles against the field produced by the potential difference across the electrodes.

The initial values of n and N are specified and since the potential difference across the gap is fixed, the quantity E must satisfy the condition

$$\int_0^1 E(x, t)\, dx = 1$$

at every instant.

It often occurs in connection with systems of conservation laws, that the solutions contain surfaces of discontinuity across which the velocity of particles is discontinuous (shock waves). Such solutions cannot exist in the system of equations considered here, since the equation for E shows that E must be continuous, even when the number densities n and N are discontinuous. However, the solutions can (and do) contain surfaces across which n or N are discontinuous. (In gas dynamics, a surface across which density is discontinuous and velocity is continuous is called a contact surface.)

One of the most effective methods available for solving these equations of motion is the method of characteristics. The paths of the negatively charged particles can be found by solving

$$(dx/dt) = E \qquad \text{(E-4)}$$

and the equation of continuity for these particles tells us that along these paths

$$(dn/dt) = sn\,(N - n) \quad . \qquad \text{(E-5)}$$

We will sometimes refer to the paths of the charged particles as a family of characteristic curves. Such a family of characteristics is shown in Figure 2E. Similarly there is a family of characteristic curves for the positively charged particles:

$$(dx/dt) = \alpha E \qquad \text{(E-6)}$$

and along these directions the conservation law for N gives the equation

$$(dN/dt) = \alpha\, s\, N\,(N - n) \quad . \qquad \text{(E-7)}$$

Figure E-3 illustrates a general scheme which can be used to integrate the equations of motion. At $t = 0$ we know $n(x, 0)$ and $N(x, 0)$; therefore, $E(x, 0)$ can be calculated. Knowing these quantities, the characteristic curves for n and N can be constructed in the neighborhood of $t = 0$ and n, N obtained along the characteristic by integrating equations (E-5) and (E-7). Once n and N are obtained at $t = \Delta t$, the electric field E can be found from Poisson's equation and the process can be repeated.

The idea of integrating along characteristic curves can also yield analytical solutions of the equations of motion.

2. The case where only one type of charged particle is present, distributed uniformly at t = 0.

Suppose that only electrons are present in the gap. Then the equations of motion are

$$(\partial n/\partial t) + (\partial/\partial x)(nE) = 0 \quad , \qquad n(x, 0) \text{ given}$$

$$(\partial E/\partial x) = sn \qquad \int_0^1 E(x, t)\, dx = 1$$

The first equation can be written in characterictic form.

$$\text{Along } (dx/dt) = E \quad , \quad (dn/dt) = -s n^2$$

with $x = x_o$, $n = 1$ at $t = 0$ (See Figure E-3.)

The characteristic equation for n can be integrated immediately:

$$n = 1/(1+st) \tag{E-8}$$

and this equation for n(x, t) holds throughout the region covered by the characteristic curves streaming from the initial curve t = 0. Outside this region we have to set n = 0, since we assume particles are not being emitted at the cathode. The situation can be seen clearly in Figure E-4.

Since the number density n(x, t) is known at each point of the gap for every instant, we can determine the field intensity E(x, t) by integrating n(x, t) with respect to x (Poisson's equation):

$$E(x, t) = \begin{cases} (d\xi/dt) & \text{if} \quad 0 \leq x \leq \xi(t) \\ \left[s/(1+st)\right](x-\xi) + (d\xi/dt) & \text{if} \quad \xi(t) \leq x \leq 1 \end{cases} \tag{E-9}$$

where $x = \xi(t)$ is the equation of the characteristic curve emanating from the origin of the (x, t)- plane. (Thus, $(d\xi/dt) = E(\xi, t)$, a condition obviously met by equation (E-9).

If the function $\xi(t)$ were known, the solution of the problem would be complete. This function can be found from the condition that the voltage drop across the gap is prescribed (even if this voltage were a prescribed function of time instead of a constant, $\xi(t)$ could still be found.) Here we encounter two possibilities:

(a) The initial number density is so large that the induced field overpowers the externally applied field and the characteristic streaming from the origin of the (x, t) plane is forced into the cathode by forces of repulsion. In this case the entire width of the gap is covered by characteristics streaming from t = 0 and it is easy to show that

$$n(x,\ t) = \left[1/(1+st)\right] \quad \text{and} \quad E(x,t) = \left[s/(1+st)\right](x-\tfrac{1}{2}) + 1 \qquad \text{(E-10)}$$

everywhere in the gap. Thus, the number density will become smaller and smaller as time passes, but remains uniform. At some instant the forces due to the applied field will overpower the forces of repulsion due to the charged particles remaining in the gap.

(b) When this happens, the characteristic streaming from the origin of the (x, t) plane begins moving across the gap towards the anode and we must have

$$\int_0^{\xi(t)} (d\xi/dt)\, dx + \int_{\xi(t)}^{1} \left\{ \left[s/(1+st)\right](x-\xi) + (d\xi/dt) \right\} dx = 1$$

since the voltage drop is prescribed. The integrations are easily performed and an ordinary differential equation for $\xi(t)$ is obtained:

$$(d\xi/dt) = 1 - \frac{(s/2)}{1+st}(1-\xi)^2 \quad , \quad \text{with } \xi(0) = 0 \quad . \qquad \text{(E-11)}$$

Notice from this last equation that if $s > 2$, $(d\xi/dt)$ is initially negative. Thus $s = 2$ is the criterion for determining when the forces of repulsion overcome the applied field. In Reference 1, Demetriades gives a solution of equation (E-11) in terms of modified Bessel functions.

The trajectories of the charged particles in the (x, t)- plane can be found from

$$(dx/dt) = E(x,\ t)$$

or, equivalently:

$$(dx/dt) - \left[s/(1+st)\right] x = (d\xi/dt) - \left[s/(1+st)\right] \xi \quad , \quad x(0) = x_o \quad .$$

3. The gap containing stationary positive ions and moving electrons.

If the particles of positive sign are massive compared to the negative charges in the gap (which is the case for electrons and ionized atoms, for example) then the mobility of the positive charges will be small compared to that of the negative charges. It is therefore interesting to study the case where $\alpha = 0$. Assume that the positive ion density throughout the gap is

$$N(x, t) = 1 \quad .$$

Then the equations of motion of the negative charges are

$$(\partial n/\partial t) + (\partial/\partial x)(nE) = 0$$

$$(\partial E/\partial x) = s(n - 1) \quad .$$

In characteristic form the first equation is

$$(dn/dt) = s\,n\,(1 - n) \quad \text{along} \quad (dx/dt) = E \quad .$$

Therefore, along the characteristic streaming from the point $(x_o, 0)$ of the (x, t)- plane we find

$$n = \frac{n(x_o, 0)\, e^{st}}{1 + (e^{st} - 1)\, n(x_o, 0)} \quad .$$

Let us again assume that the particles are distributed uniformly. Then $n(x_o, 0) = n_o$ (a constant) and

$$n = \frac{n_o\, e^{st}}{1 + (e^{st} - 1)\, n_o} \quad . \qquad \text{(E-12)}$$

Notice that if the initial electron number density is equal to the initial ion number density then $n_o = 1$ and $n(x_o\, t) = 1$.

In any case, the electric field can be found by integrating with respect to x:

$$E = \begin{cases} \dfrac{(n_o - 1)}{1 + (e^{st} - 1)\, n_o}\,(x - \xi) + (d\xi/dt) & \text{if} \quad 0 \leq x \leq \xi(t) \\ -\,s\,(x - \xi) + (d\xi/dt) & \text{if} \quad \xi(t) \leq x \leq 1 \end{cases} \qquad \text{(E-13)}$$

where $x = \xi(t)$ is once again the equation of the characteristic streaming from the origin of the (x, t) plane (See Figure E-5.).

To complete the solution it is necessary to find the function $\xi(t)$, and we again take recourse to the fact that the voltage drop across the gap is prescribed. As before, there are two possibilities:

(a) There are so many electrons in the gap initially that electrons are forced out of the gap through both electrodes because of repulsive forces. In this case the solution is

$$n(x, t) = \frac{n_o e^{st}}{1 + (e^{st} - 1) n_o} , \quad E(x, t) = \frac{s (n_o - 1)}{1 + (e^{st} - 1) n_o} (x - \tfrac{1}{2}) + 1 .$$

In this case, as $t \longrightarrow \infty$ the electron number density approaches unity. It will be shown however that when the electron density n_o becomes equal to or greater than $1 + (2/s)$, something else happens.

(b) If the number of electrons in the gap is sufficiently small, all the electrons, and in particular the electron starting at the origin of the $(x_o t)$- plane, begin moving across the gap toward the anode. In this case

$$\int_0^{\xi(t)} E(x, t)\, dx + \int_{\xi(t)}^{1} E(x, t)\, dx = 1$$

or

$$(d\xi/dt) + (s/2)\ \xi^2 + (s/2)\ \frac{(n_o - 1)}{1 + (e^{st} - 1) n_o} (1 - \xi)^2 = 1 , \ \xi(0) = 0 \quad \text{(E-14)}$$

The function $\xi(t)$ can be found from this ordinary differential equation and the solution is completed by substituting for $\xi(t)$ in equations (E-13). Notice that if $n_o > 1 + (2/s)$, (ds/dt) is initially negative, which proves an assertion made previously.

In the case where the number density of electrons is initially equal to the number density of ions, the solution of the equation for $\xi(t)$ is especially simple. Thus, in equation (E-14) set $n_o = 1$. Then

$$\xi = \sqrt{(2/s)}\ \tanh\ \sqrt{(s/2)}\ \ t \quad .$$

Now as $t \longrightarrow \infty$, $\xi \longrightarrow \sqrt{(2/s)}$, and therefore if $s > 2$ the gap will never clear itself of electrons and the contact surface across which the electron number density jumps will ultimately come to rest at the point $x = \sqrt{(2/s)}$.

4. Solutions of the General Problem Obtained by Numerical Integration.

The analytical solutions that have been obtained are made possible by two assumptions:

(a) Spatially uniform initial distribution of charges

(b) The motion of only one type of charged particle has to be determined.

Even when these conditions are not met it is still convenient to use the method of characteristics to find numerical solutions. The basis of such solutions are the equations (E-3), (E-4), (E-5), (E-6), and (E-7). This technique of solution has been programmed for the 7090 computer and some of the results obtained in this way will be presented here.

(a) The case where only one type of particle is present and where the number density is initially uniform. Analytical solutions for this case have already been presented and discussed. Figure E-6 illustrates the difference between the behavior of the charged particles when $s < 2$ and $s > 2$. The lower curve of Figure E-6 shows the position of the discontinuity in number density as a function of time for $s = 1$. The upper curve in the figure is for $s = 3$. The discontinuity does not leave the cathode until the number density is reduced to the point where $sn = 2$. Figures E-7 and E-8 show number density distributions and field intensity distributions at several instants of time for $s = 3$. Electrode currents as a function of time are shown in Figure E-9.

(b) The case where the gap contains monatomic hydrogen ions and electrons ($\alpha = -.023$). The non-dimensional number densities of ions and electrons were assumed to be equal. Figures E-10 and E-11 show the effect of changing s from 1 to 3. Analytical solutions were discussed for the case where $\alpha = 0.0$ and it is seen from Figures E-10 and E-11 that this simple theory gives a good picture of what happens when α is small and both types of ions are distributed uniformly at the initial instant. Figure E-10 shows a triangular region in which charge neutrality persists for some time. Within this region the number densities are $n = 1$ and $N = 1$. Outside this region there is a slight variation of number density. Figures E-12, E-13, and E-14 show the field intensity, the anode current, and the cathode current.

(c) Another interesting case is the case where the gap contains singly ionized positive atoms and atoms with an attached electron, $\alpha = -1.0$. Figure E-15 shows the effect of increasing s upon the motion of the number density discontinuities. As s is increased the charged particles in the middle portion of the gap (neutral region) tend to move ever more slowly, and the time required to clear the gap of charges increases almost linearly with s. Figures E-16, E-17, E-18 show number densities, field intensities, and anode current as a function of time (anode current and cathode current have identical time histories because of the symmetry of the problem when $\alpha = -1.0$). All of these calculations were made assuming that the charged particles were distributed uniformly initially, and that the positive charge density was the same as the negative charge density initially (initial charge neutrality).

The digital program is capable of handling problems with arbitrary initial charge distributions, and with relatively minor modifications could be adapted so as to include recombination processes and ionization processes (such processes result in production terms on the right side of the equations of conservation for n and N).

5. Conclusions

The problem of calculating the motion of charges under the action of electric fields produced by the charges themselves is a non-linear problem.

It has been shown how the method of characteristics can be used to find analytical solutions to such problems. Practically all of the analytical results presented in this appendix have been obtained previously by Demetriades and by Morrison and Edelson. Demetriades uses Monge's method, and Morrison and Edelson use Charpit's method.

The numerical solutions were found by mechanizing the method of characteristics for the 7090 computer. Computing times for the solutions presented ranged from 8 seconds to several minutes. The FORTRAN program is easily modified to include charge production mechanisms such as ionization and recombination.

The method of characteristics can also be used to find analytical and numerical solutions when the inertia of the charged particles is included in the analysis. Diffusion effects can also be included in numerical schemes using the method of characteristics.

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2. Morrison, J. A. and D. Edelson, "Solution of the Space Charge Problem for a Pulsed Townsend Discharge", Journal of Applied Physics, Vol. 33, pp. 1714-1720, May, 1962.

3. Courant-Hilbert, "Methods of Mathematical Physics", Volume II.

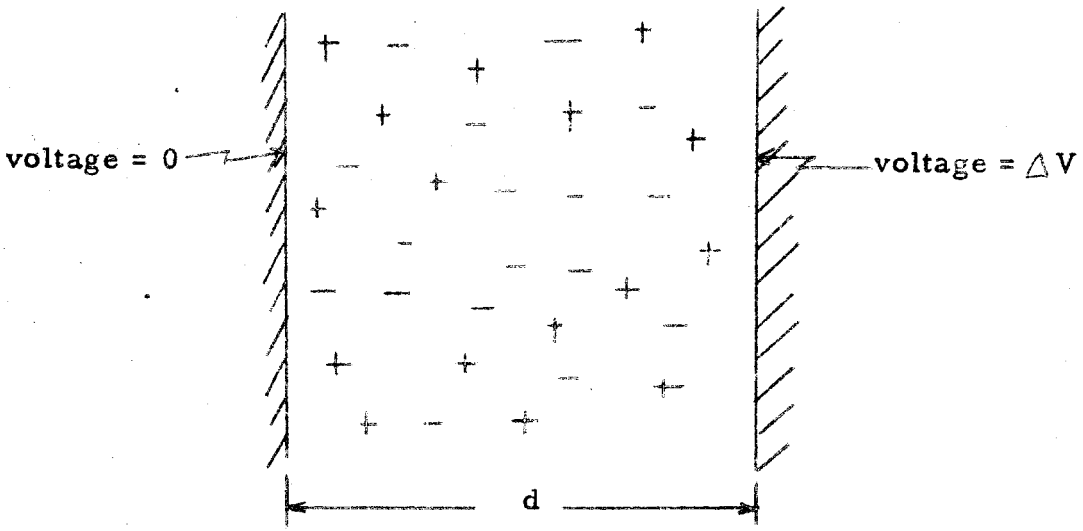


FIGURE E-1

THE PHYSICAL SYSTEM

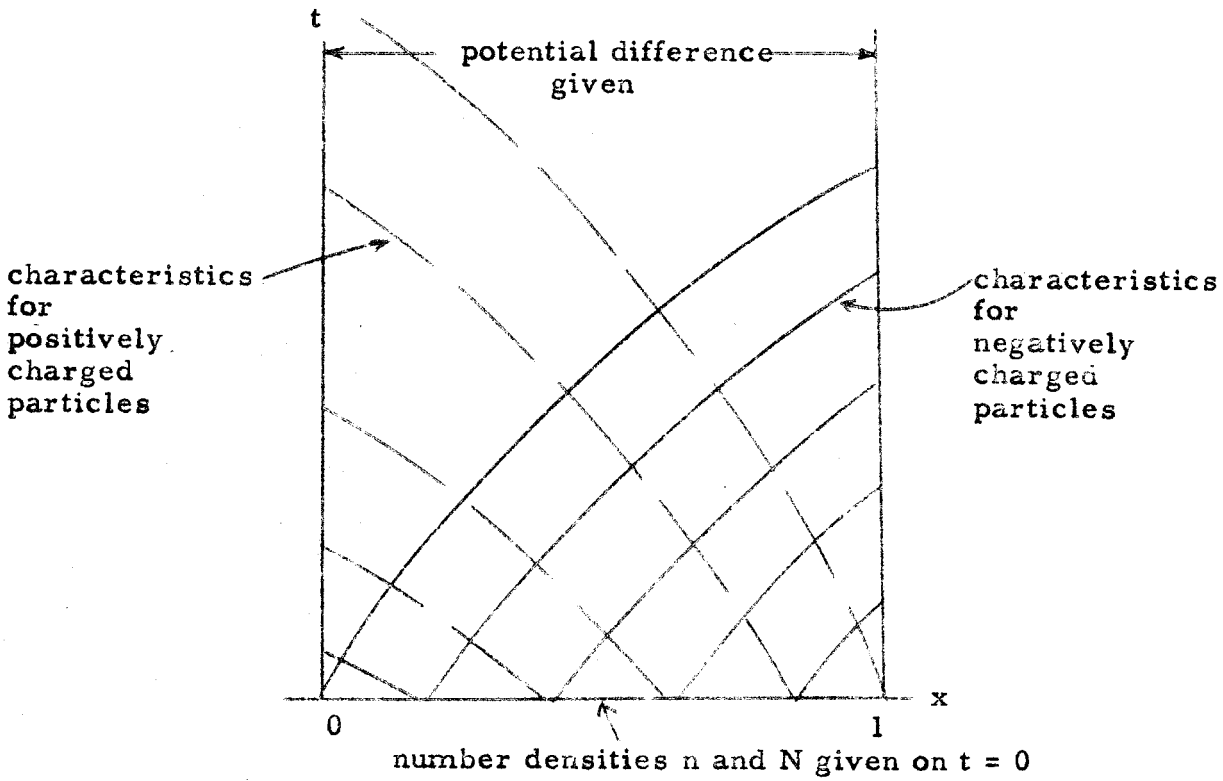
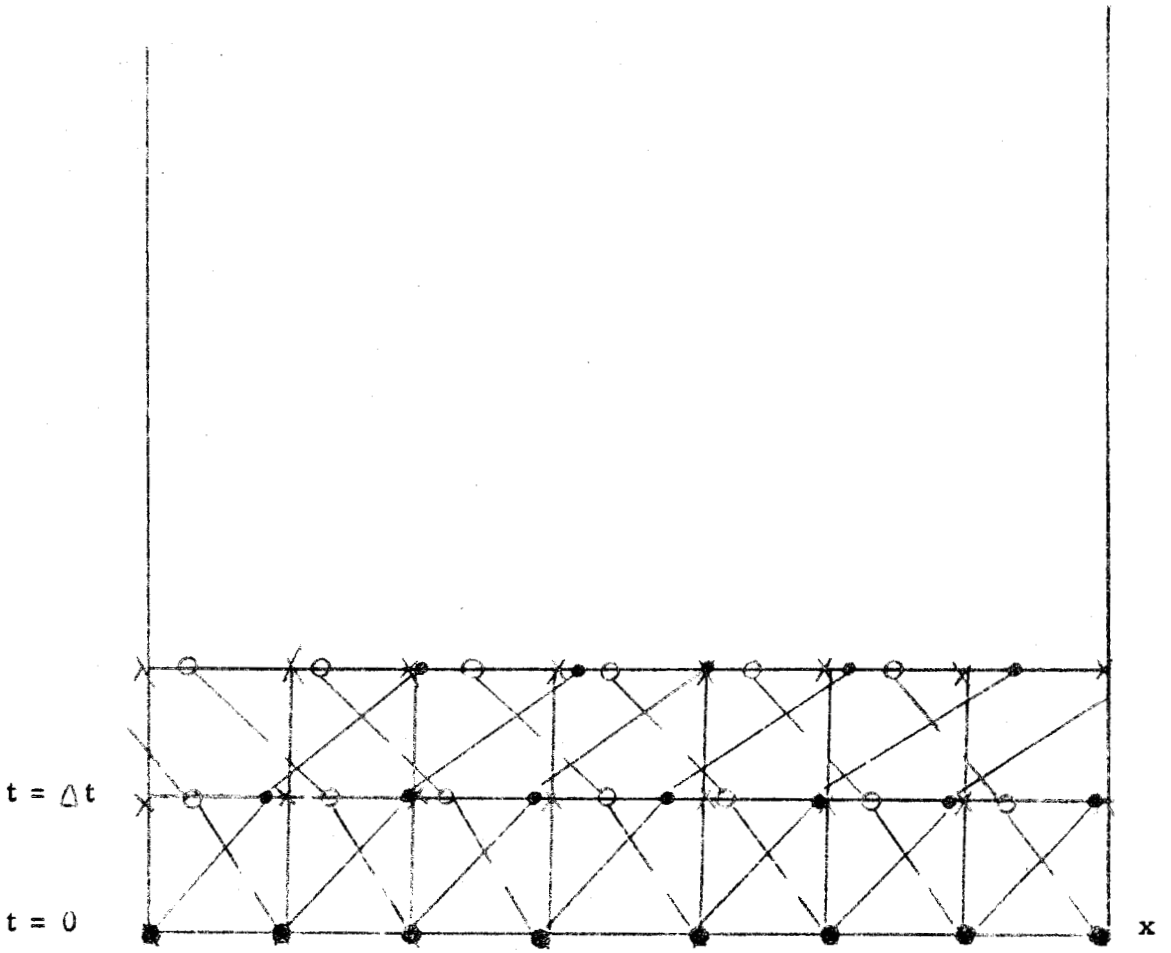


FIGURE E-2

CHARACTERISTIC CURVES IN THE (x, t)- PLANE



grid points

• points at which n can be directly calculated from $(dn/dt) = sn(N-n)$

○ points at which N can be directly calculated from $(dN/dt) = \alpha sN(N-n)$

FIGURE E-3

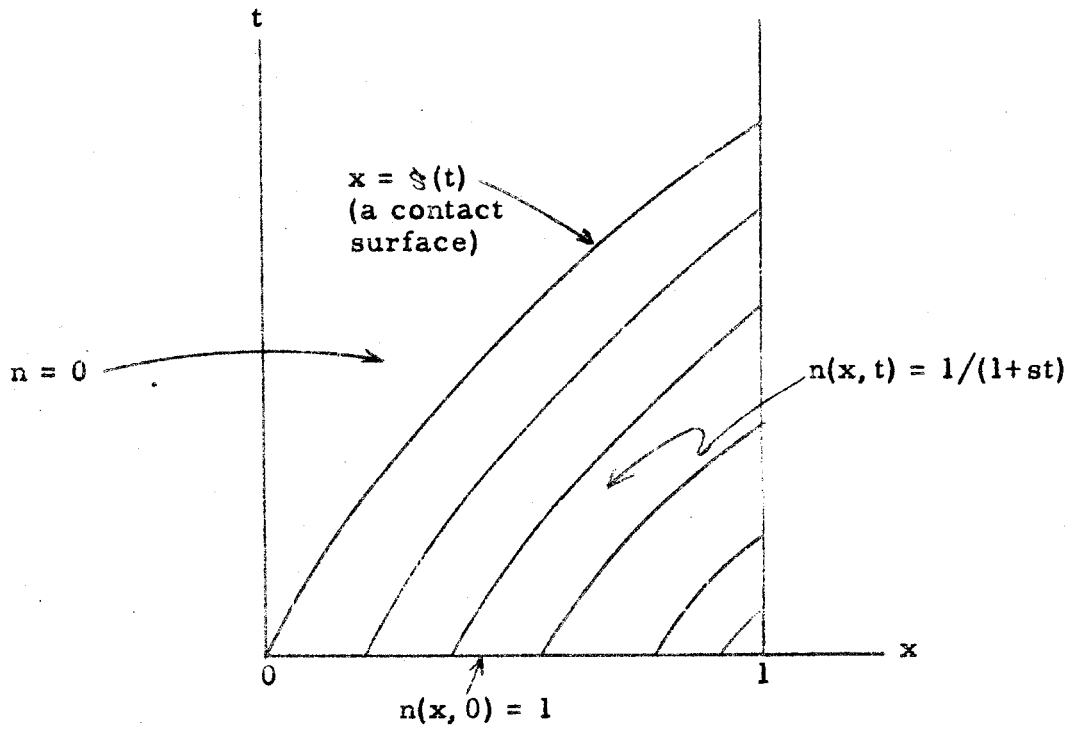


FIGURE E-4.

INITIAL VALUE PROBLEM FOR A SINGLE TYPE OF CHARGED PARTICLE

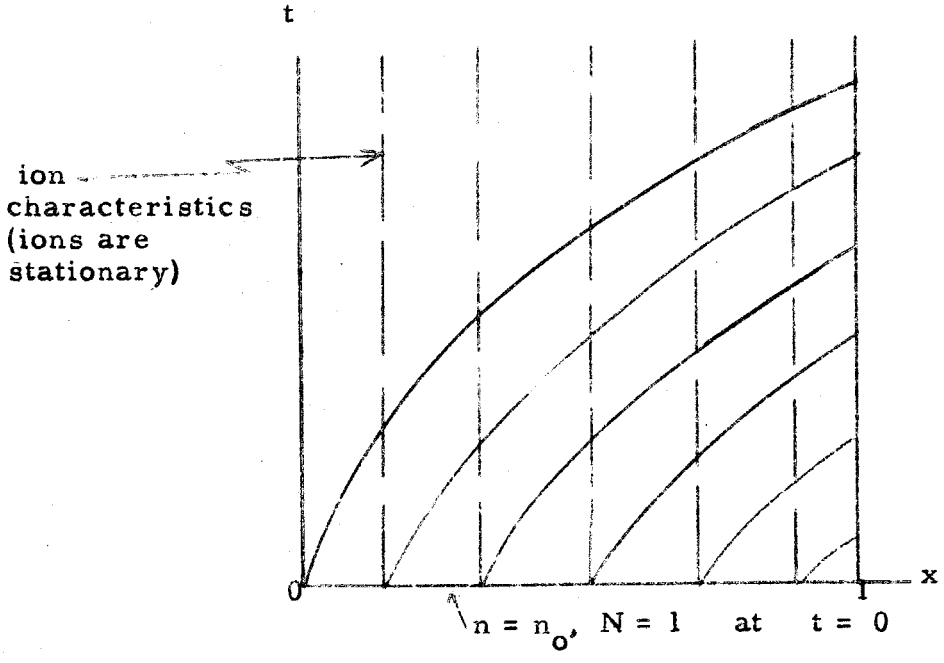


FIGURE E-5

INITIAL VALUE PROBLEM

FOR ELECTRONS MOVING IN A STATIONARY FIELD OF POSITIVE IONS

FIGURE 6E
MOTION OF DISCONTINUITIES
IN ELECTRON DENSITY
(NO IONS)

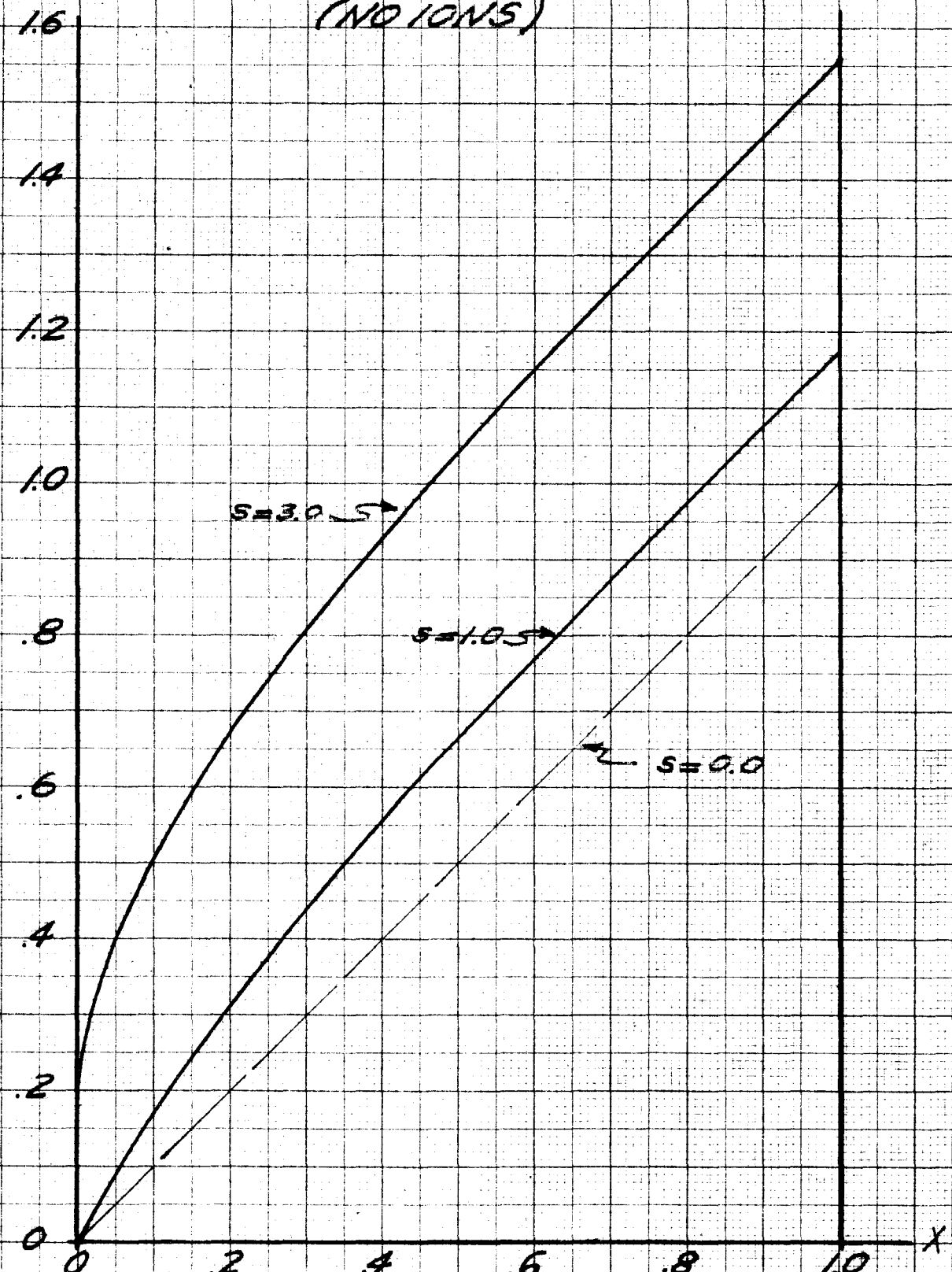


FIGURE 7E
NUMBER DENSITY

NO IONS
S=3.0

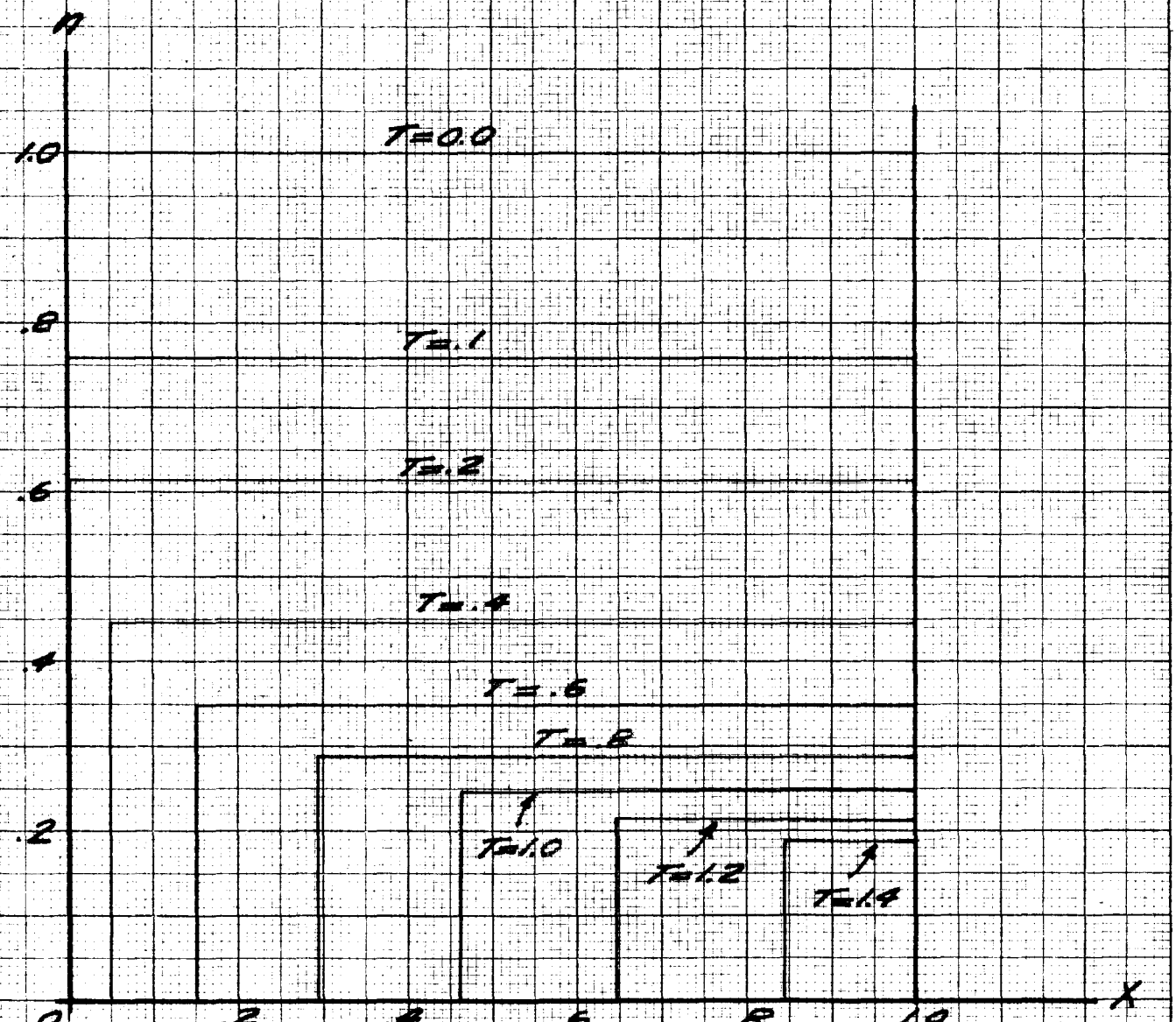


FIGURE 8E
FIELD INTENSITY DISTRIBUTIONS

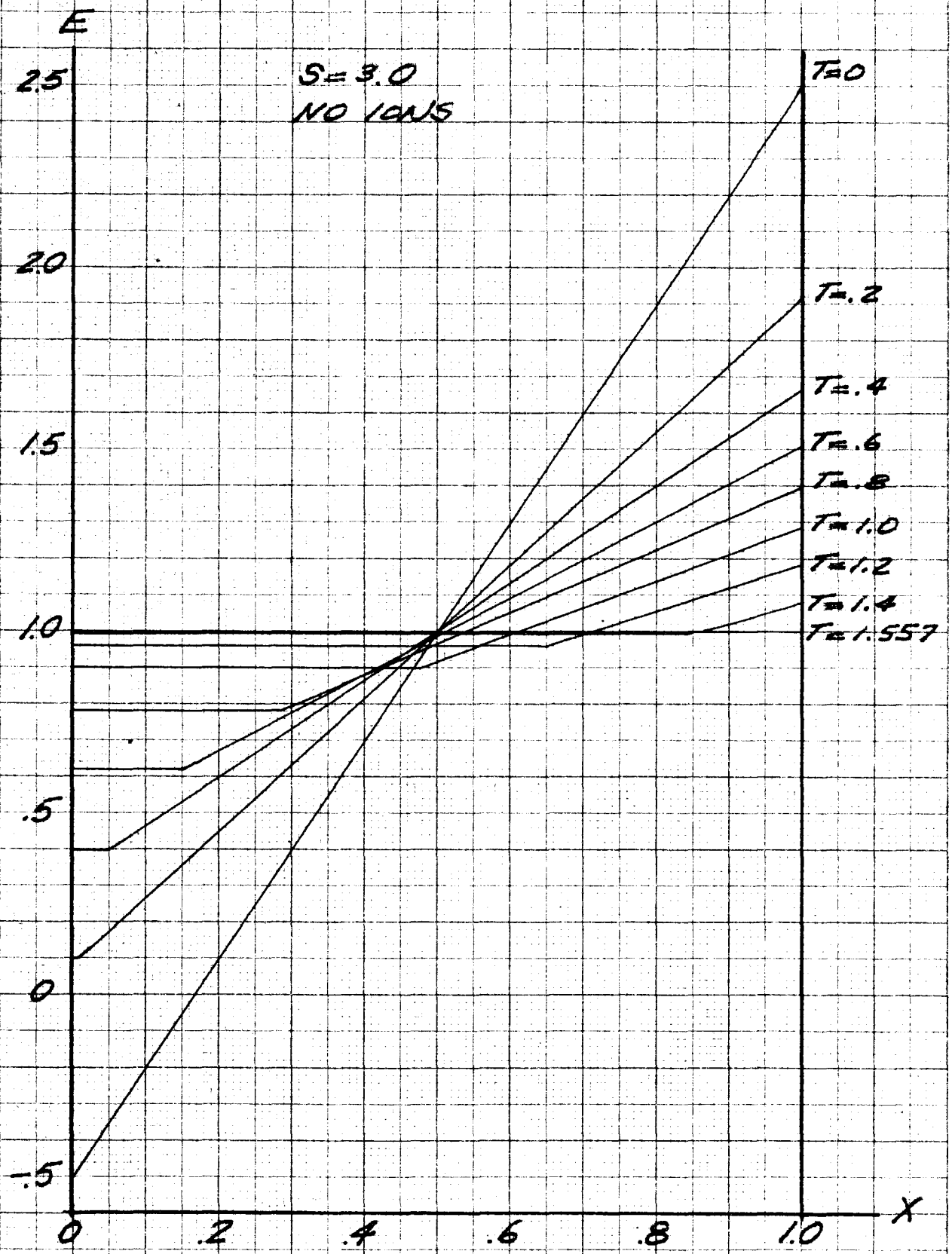


FIGURE 9E
ELECTRODE CURRENTS AS FUNCTIONS OF TIME

S=3.0
NO IONS

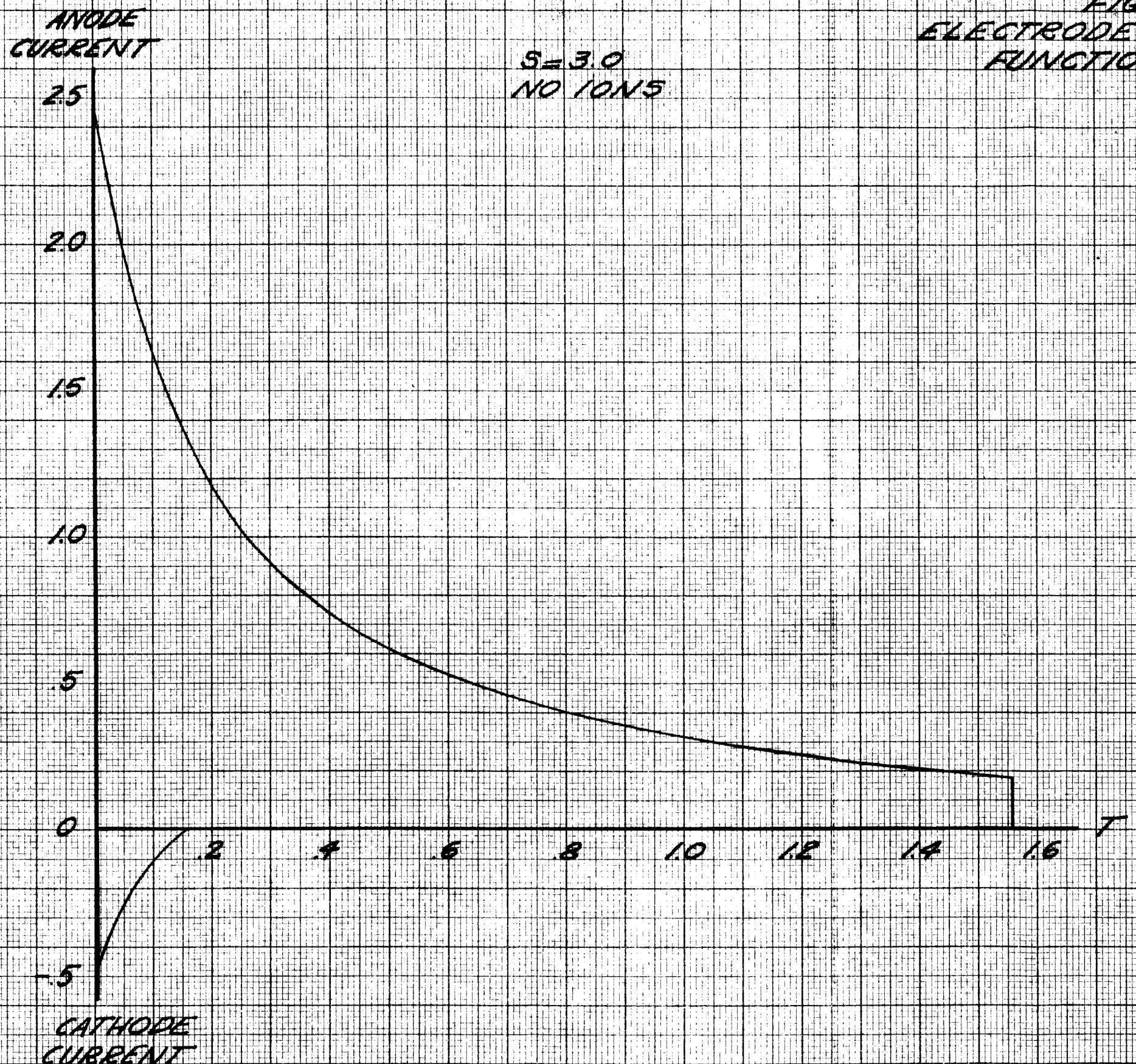
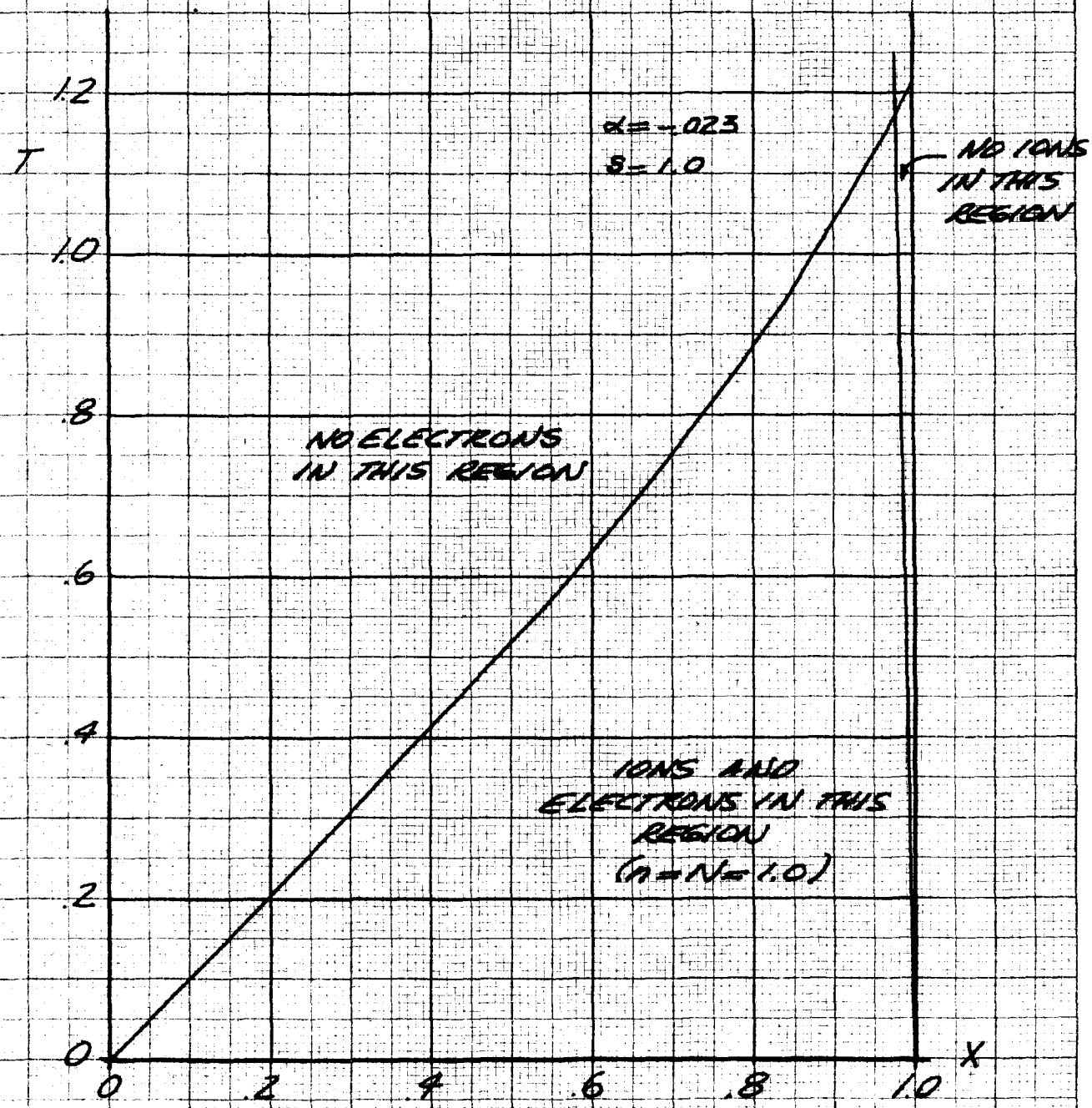


FIGURE 10E
MOTION OF CHARGE DISCONTINUITIES



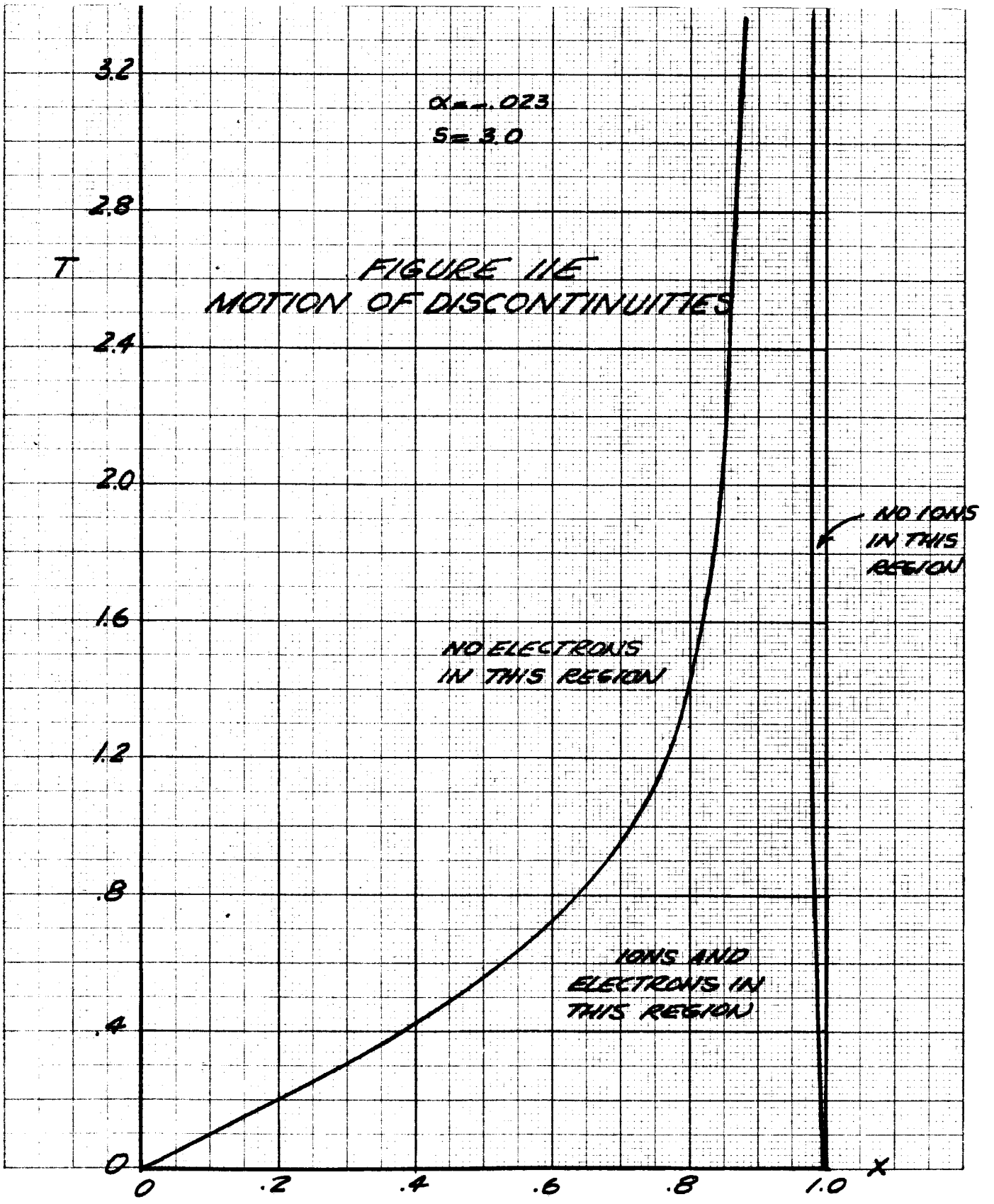

α = -.023
S = 3.0
FIGURE IIE
MOTION OF DISCONTINUITIES
T
X
3.2
2.8
2.4
2.0
1.6
1.2
.8
.4
0
.2
.4
.6
.8
1.0
NO IONS IN THIS REGION
NO ELECTRONS IN THIS REGION
IONS AND ELECTRONS IN THIS REGION

FIGURE 12E
FIELD INTENSITIES

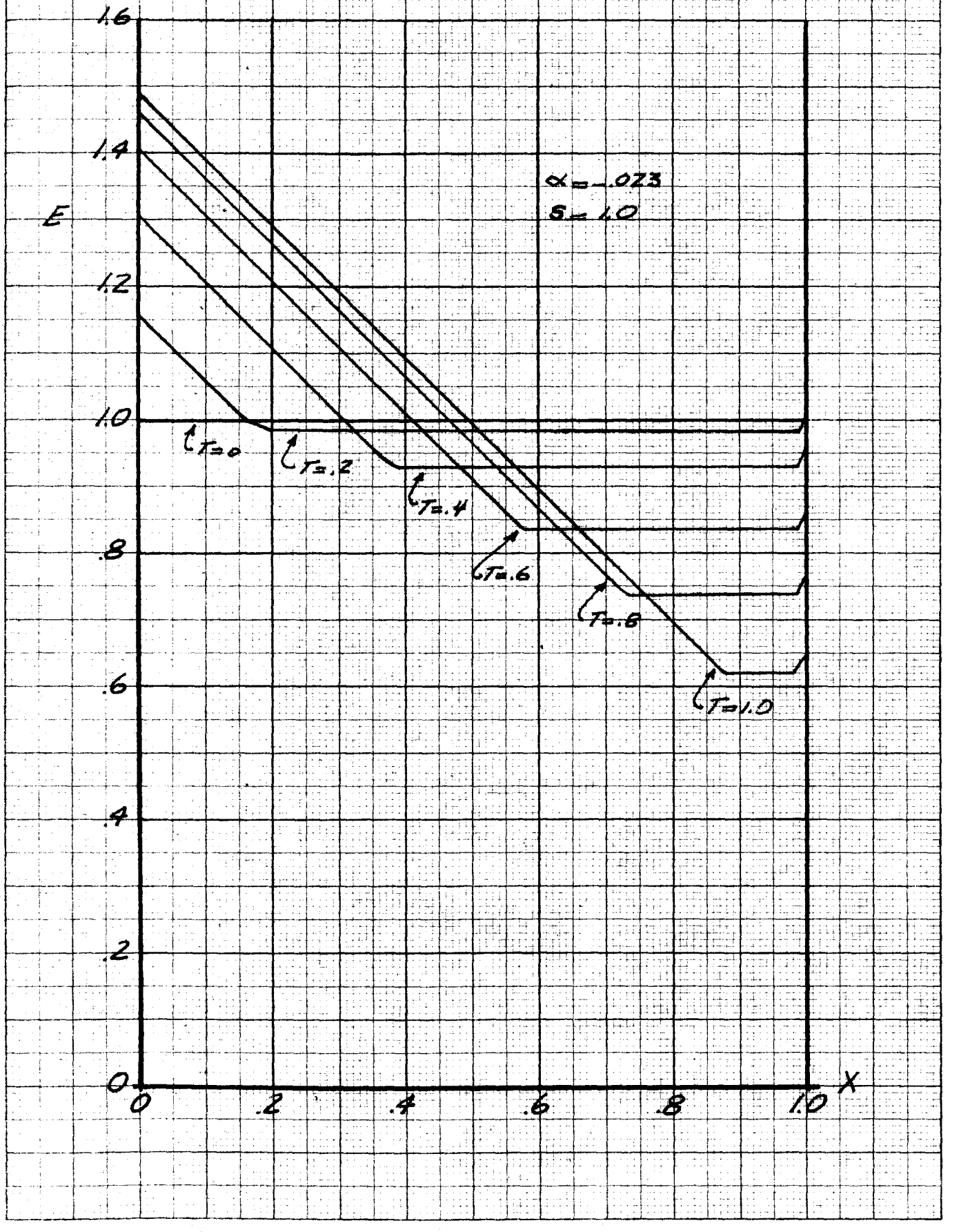
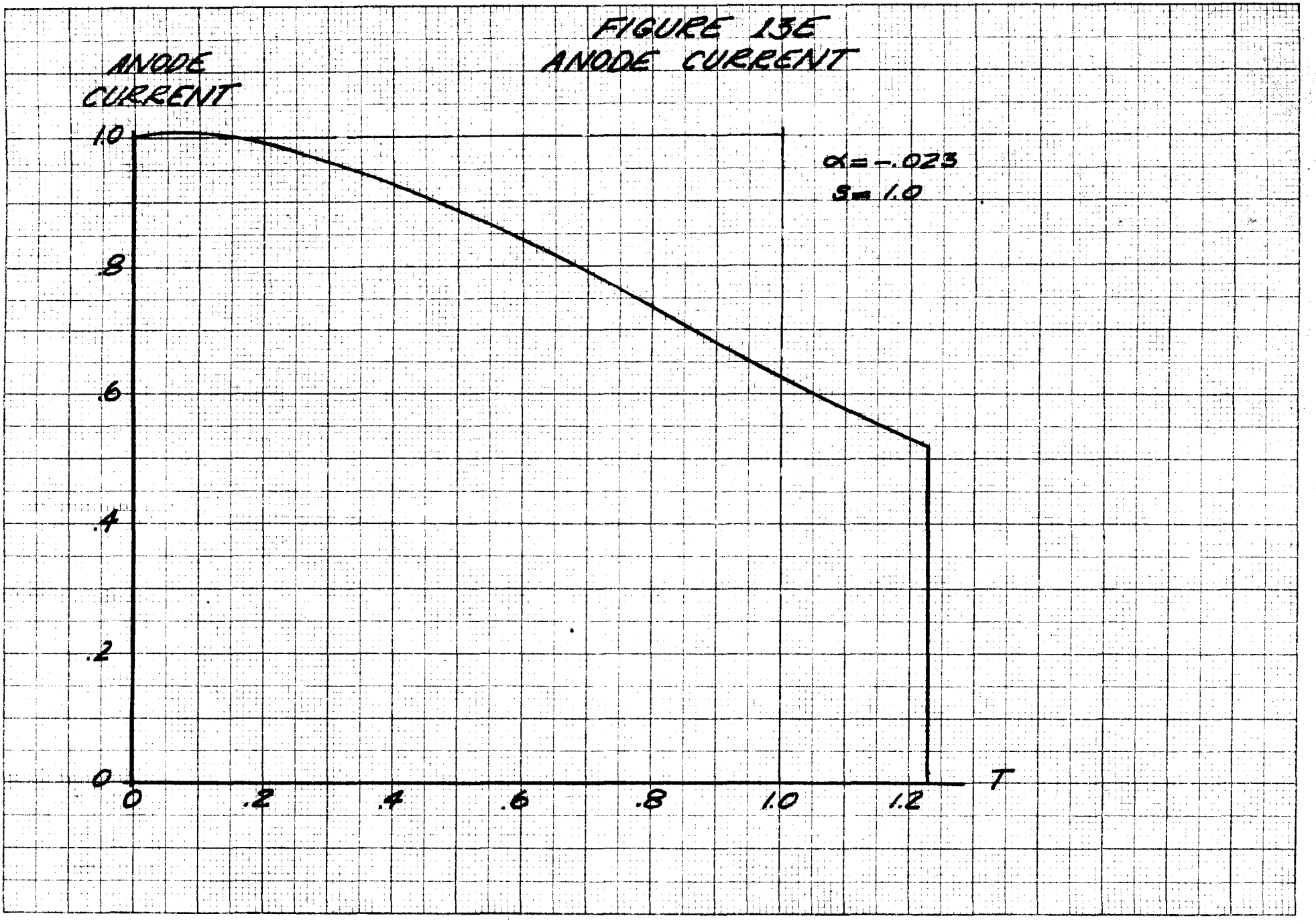


FIGURE 13E
ANODE CURRENT



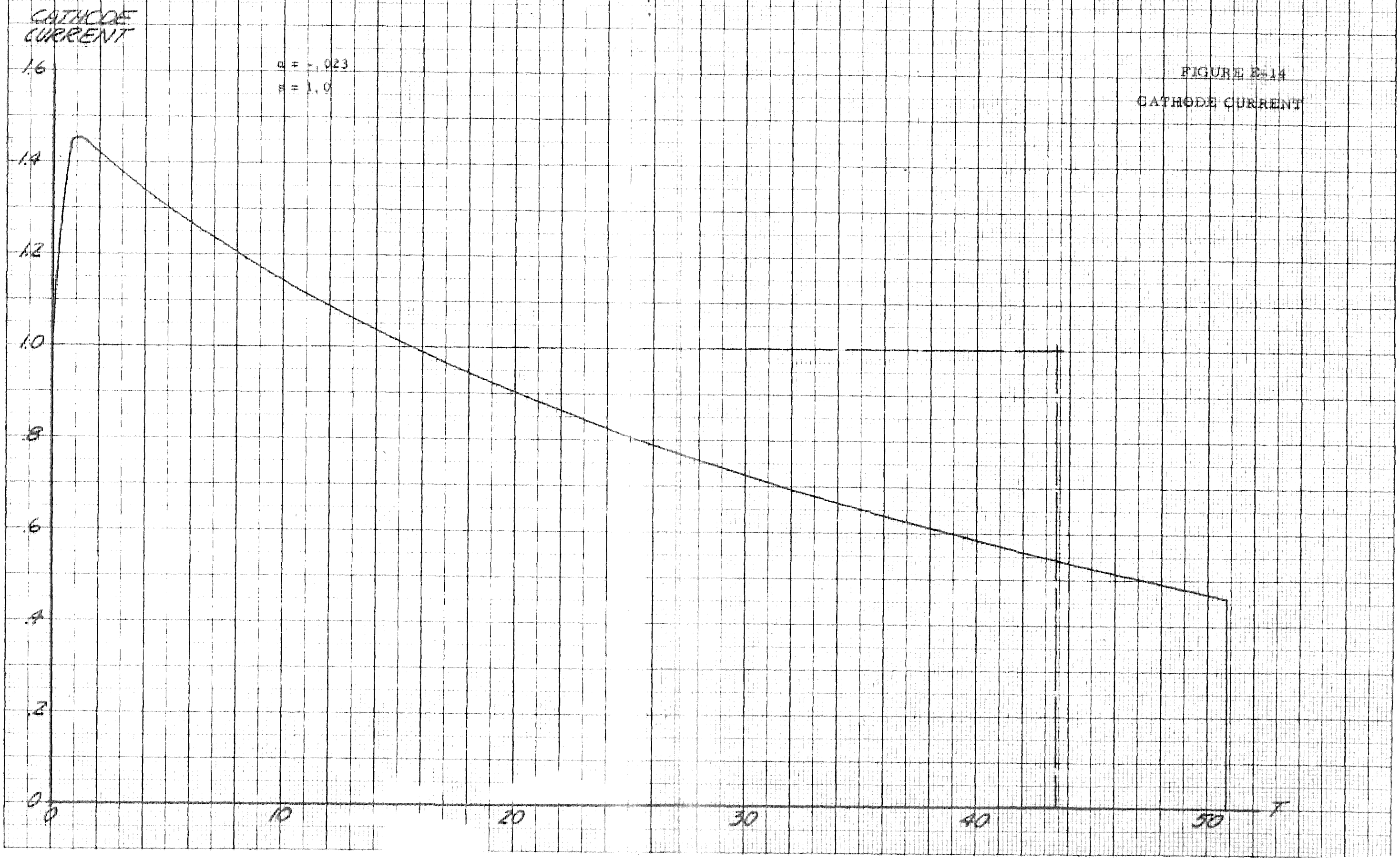


FIGURE B-14

CATHODE CURRENT

FIGURE 15E
EFFECT OF S UPON MOTION OF CHARGE DISCONTINUITIES

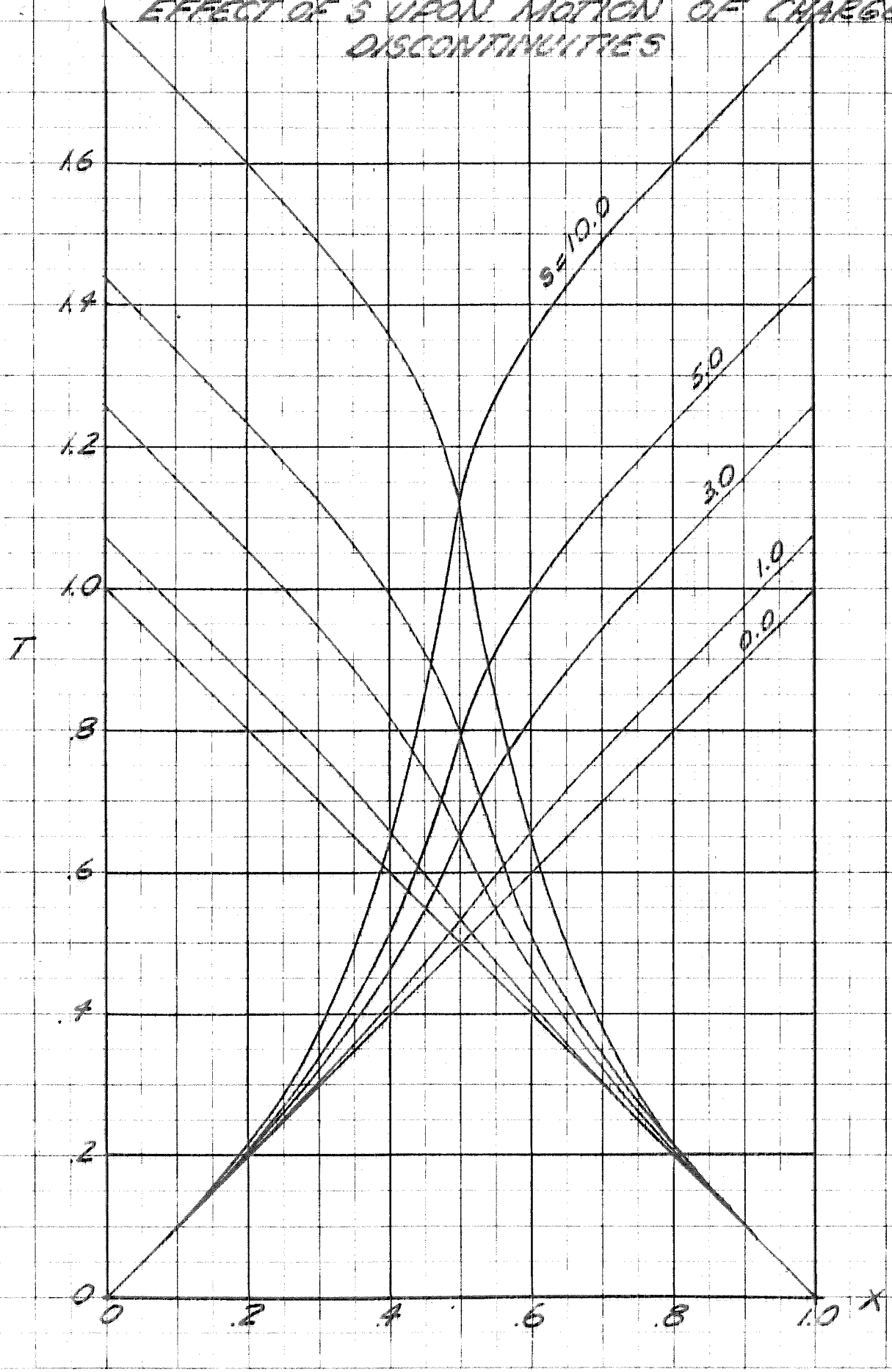


FIGURE 16E
NUMBER DENSITY OF ELECTRONS

$\alpha = -1$

$S = 3$

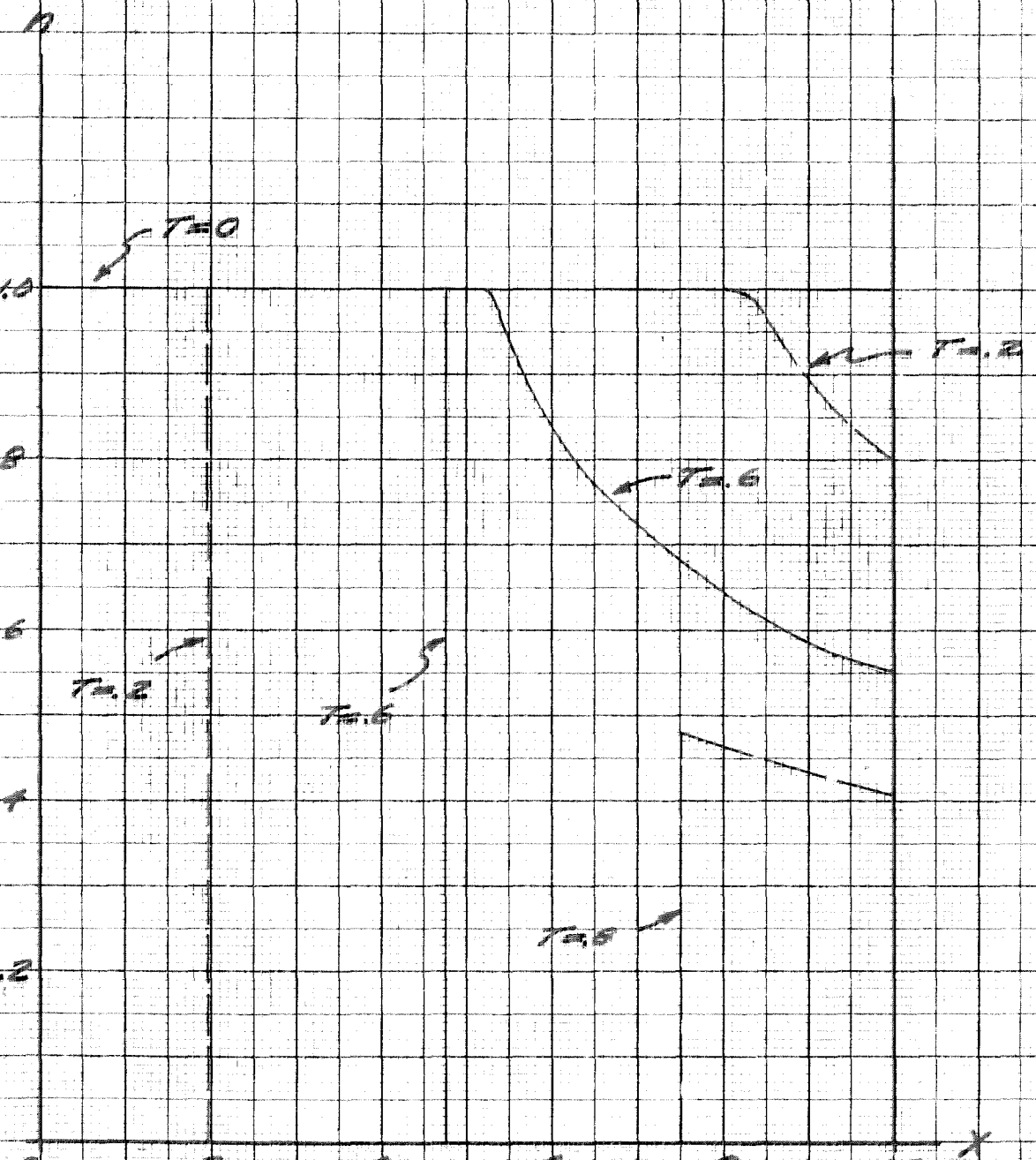
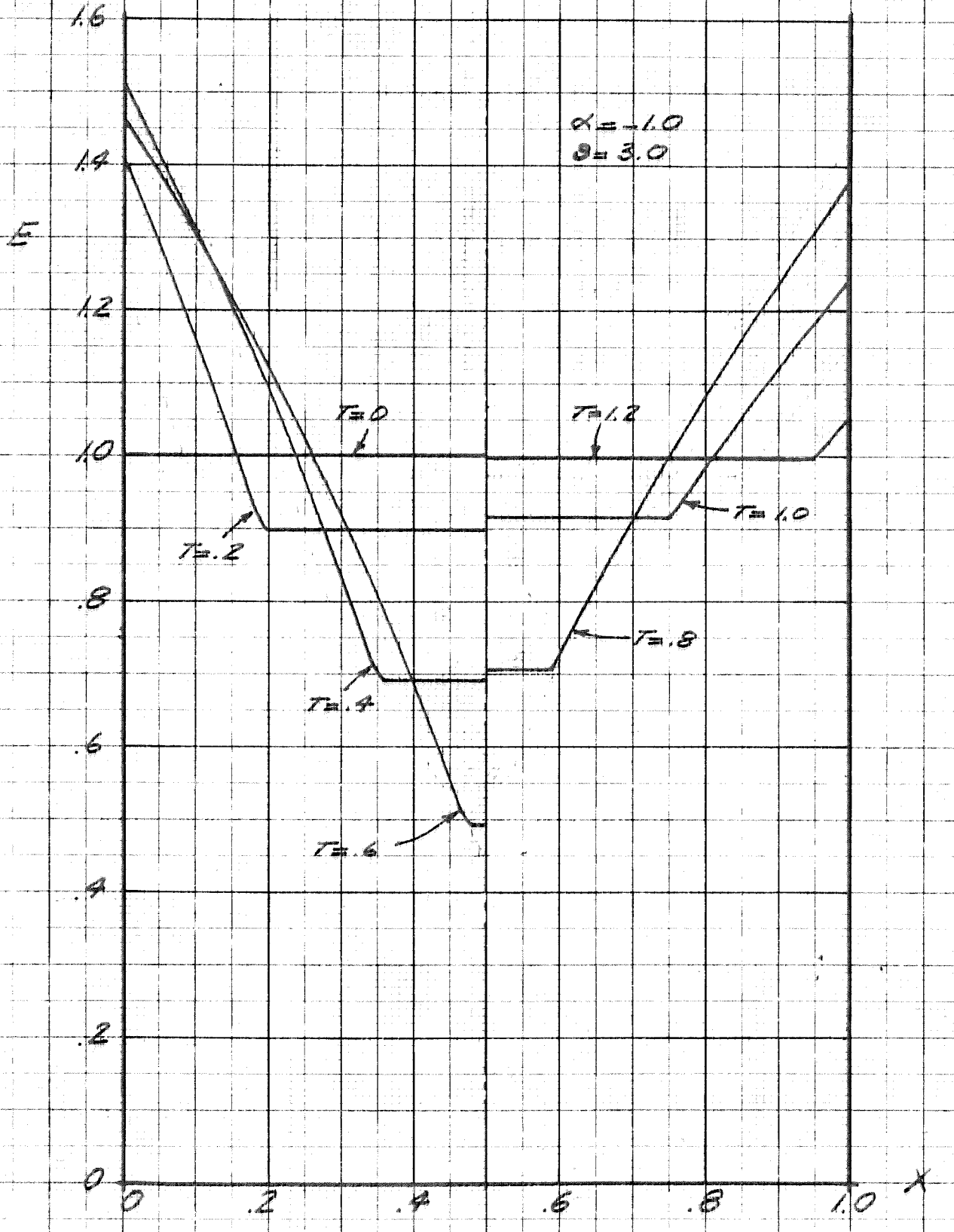


FIGURE 17E
FIELD INTENSITIES



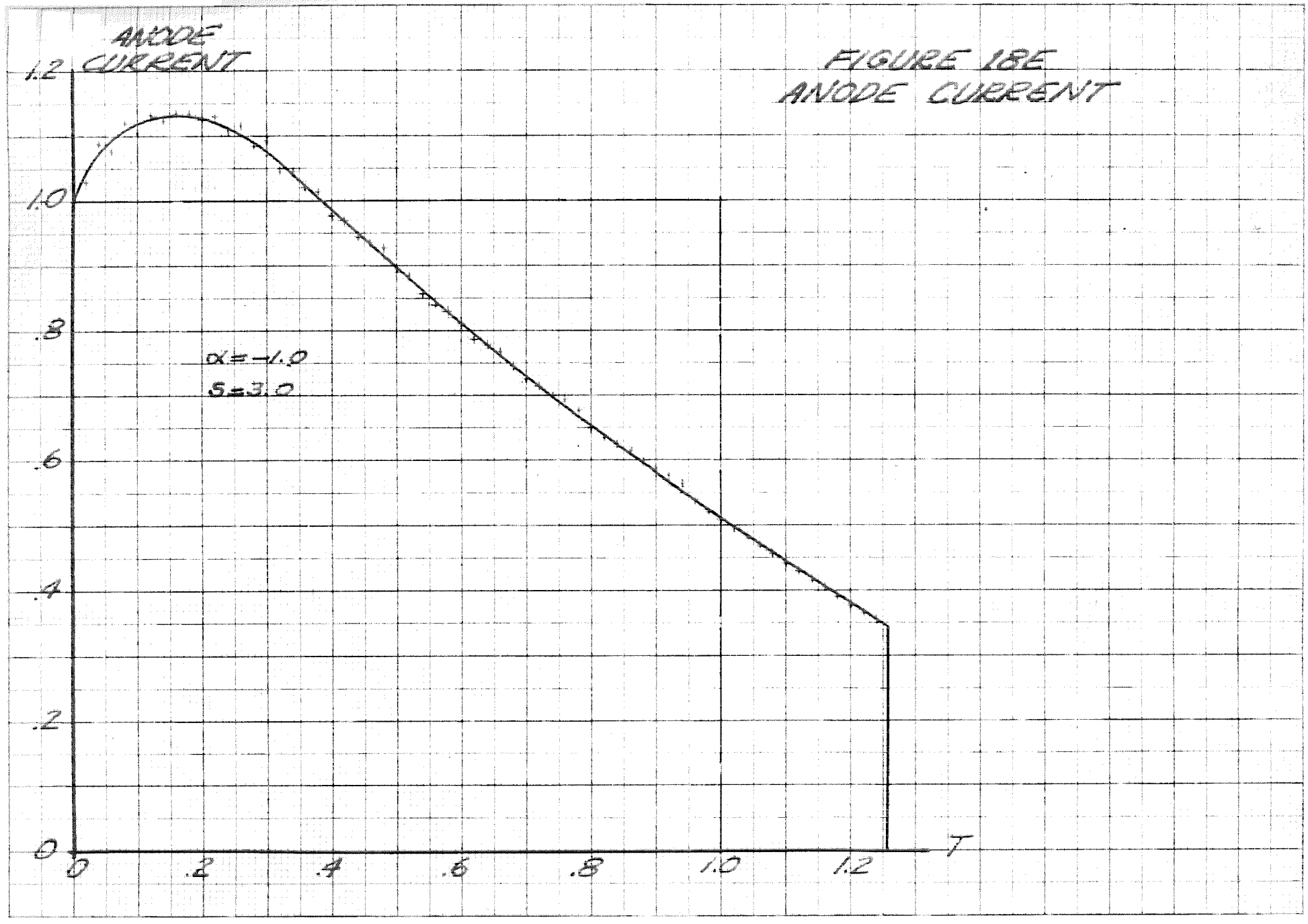


FIGURE 18E
ANODE CURRENT